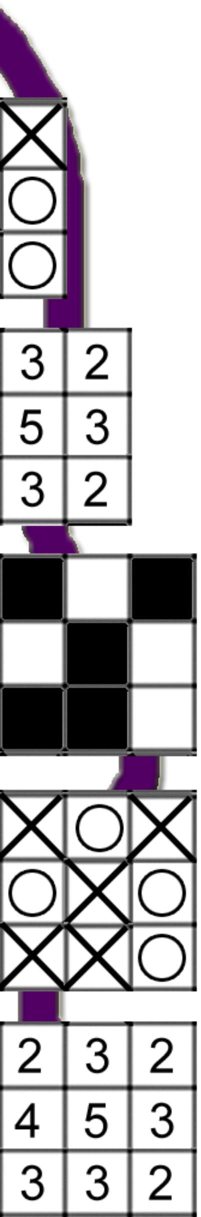


Think-Tac-Toe: When are puzzles solvable?

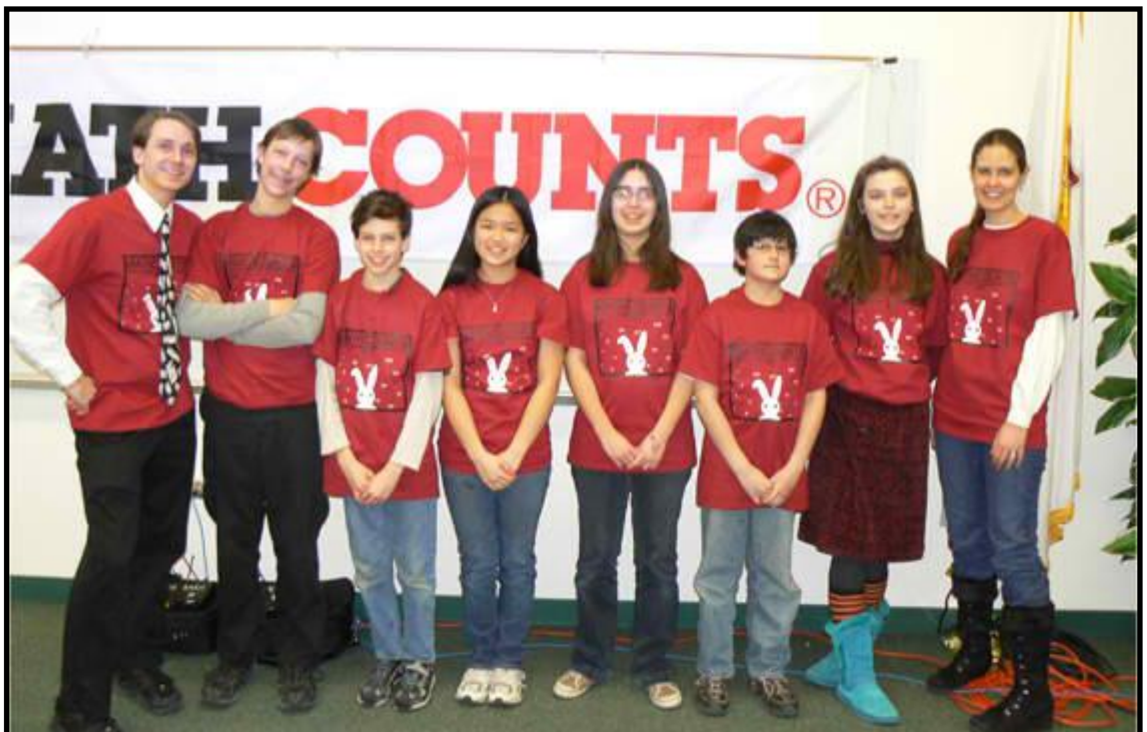
Susa Stonedahl & Forrest Stonedahl

MathFest: August 5, 2011

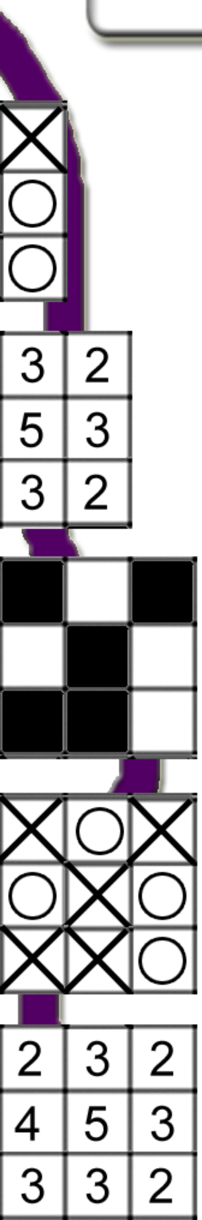
Motivation

Co-taught a middle school math enrichment program

- Students like puzzles
- Experienced and inexperienced mathematicians are on more even ground when facing a new puzzle



Rules of Think-Tac-Toe



- In Think-Tac-Toe the puzzler tries to discover the hidden locations of X’s and O’s in a grid by using number clues.

- The number in each square tells you the number of X’s in that square’s neighborhood.
  - A square’s neighborhood is made up of the square itself and any squares that it shares an edge or a corner with.

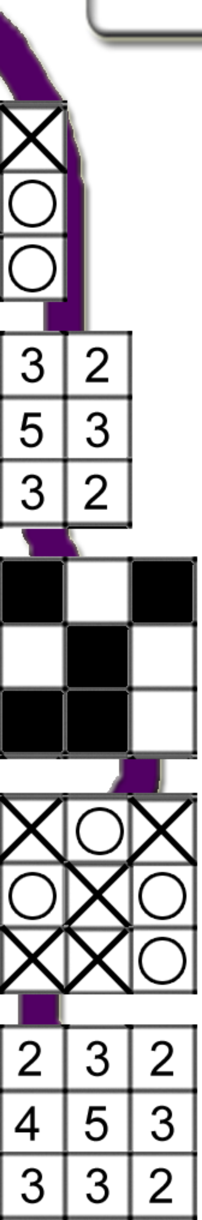
For example…

Clues

3	5	4
3	6	5
1	3	3

Solution

For example...



Clues

3	5	4
3	6	5
1	3	3

Solution

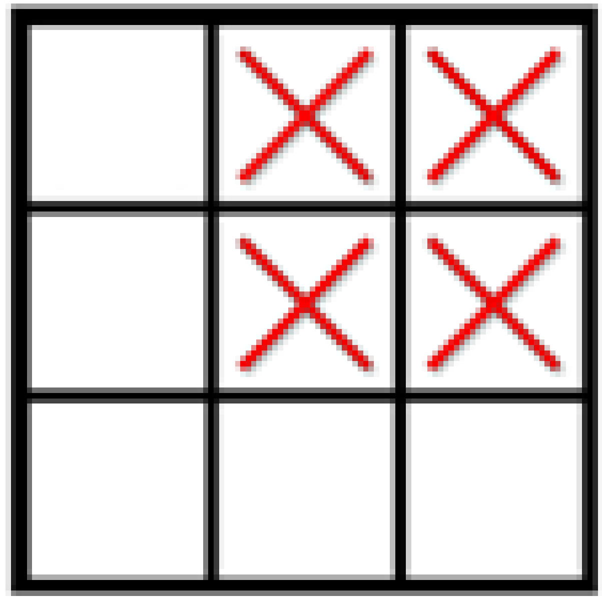
If we look at the 4....

For example…

Clues

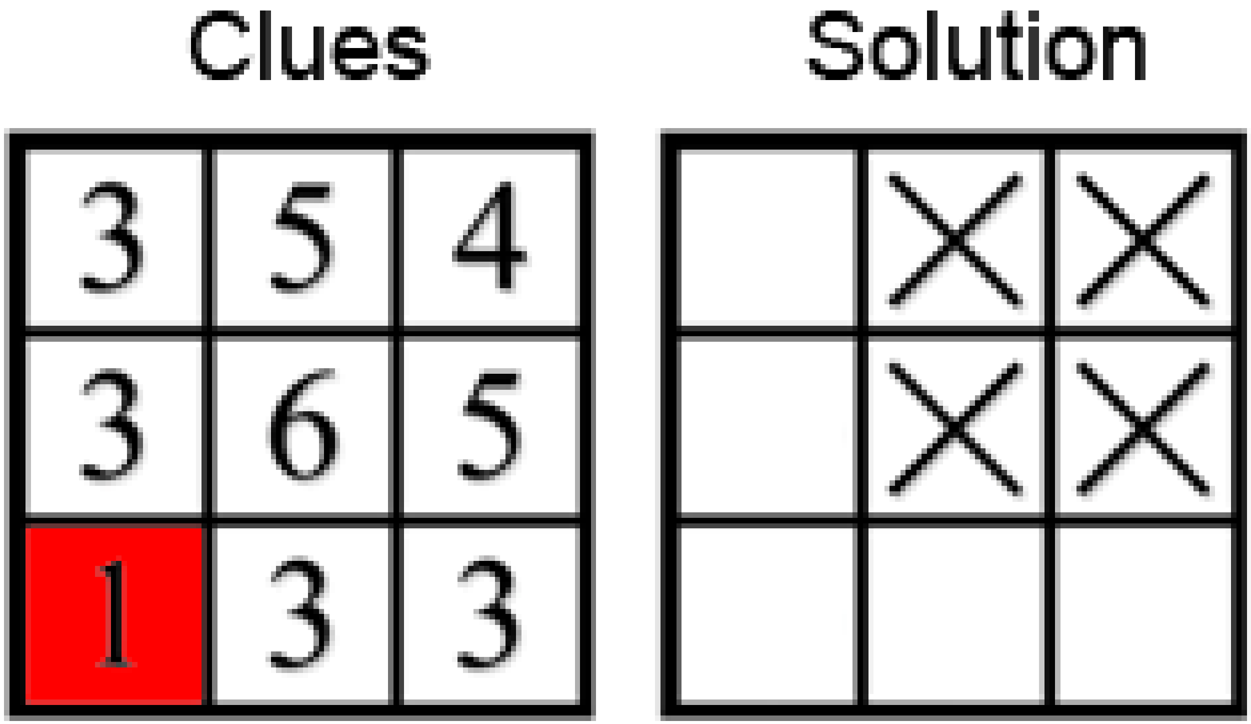
3	5	4
3	6	5
1	3	3

Solution



The whole neighborhood has X's

For example…



If we look at the 1…

For example...

Clues

3	5	4
3	6	5
1	3	3

Solution

	X	X
O	X	X
O	O	

The neighborhood already has it's "1"

For example…

Clues

3	5	4
3	6	5
1	3	3

Solution

	X	X
O	X	X
O	O	

If we look at these 3's…

For example…

Clues

3	5	4
3	6	5
1	3	3

Solution

X	X	X
O	X	X
O	O	X

They each need another X.

For example…

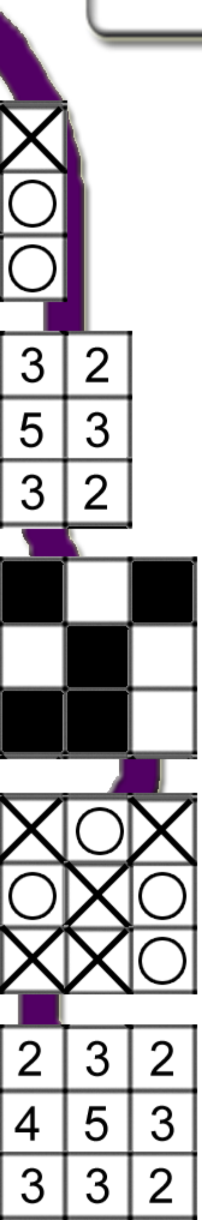
Clues

3	5	4
3	6	5
1	3	3

Solution

X	X	X
O	X	X
O	O	X

The puzzle is solved!

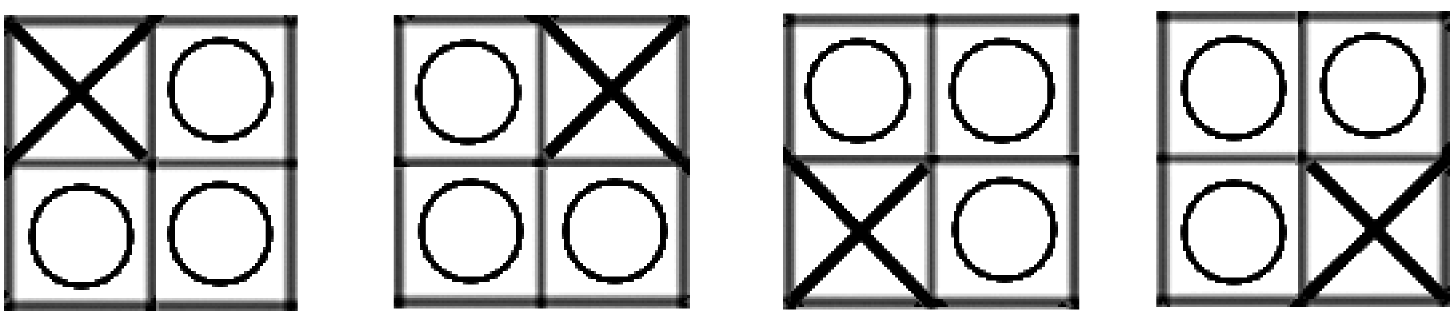


But NOT All Puzzles Are Solvable…

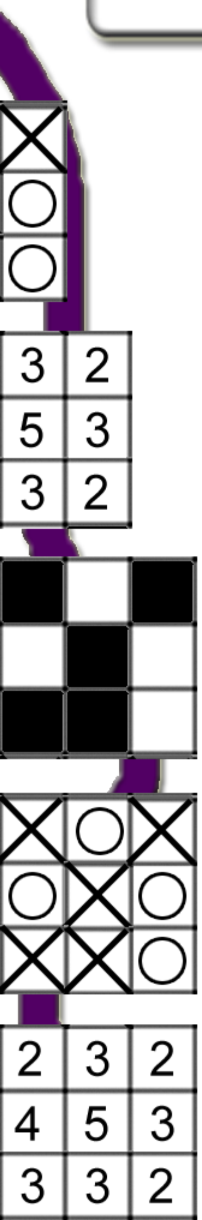
1	1
1	1

This puzzle could have originated as any of these four solutions, so it's not solvable.

(Puzzle creation operation is not invertible!)

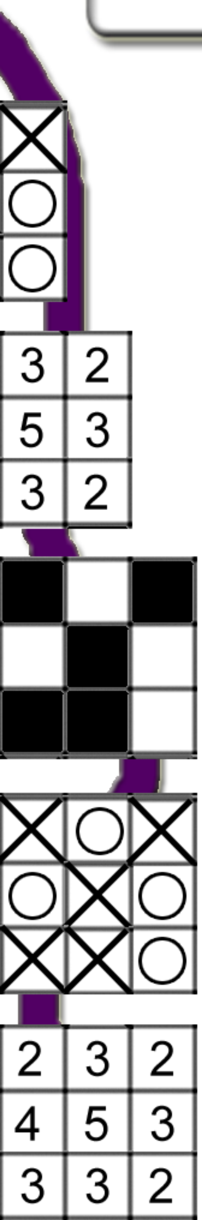


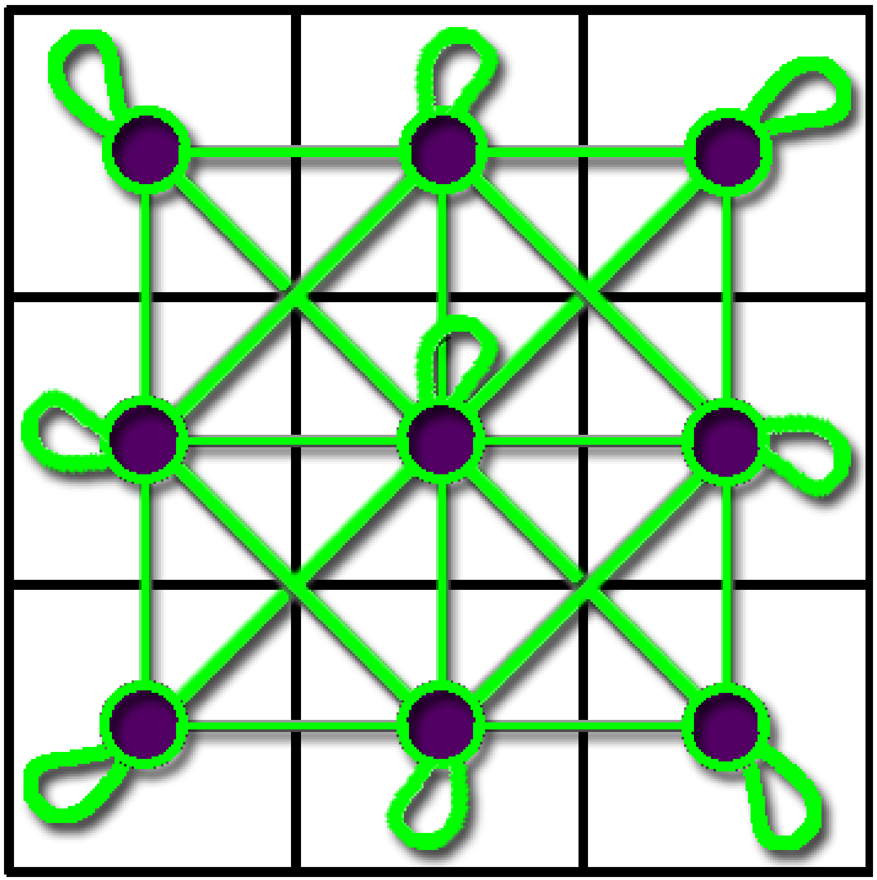
Our Question is…



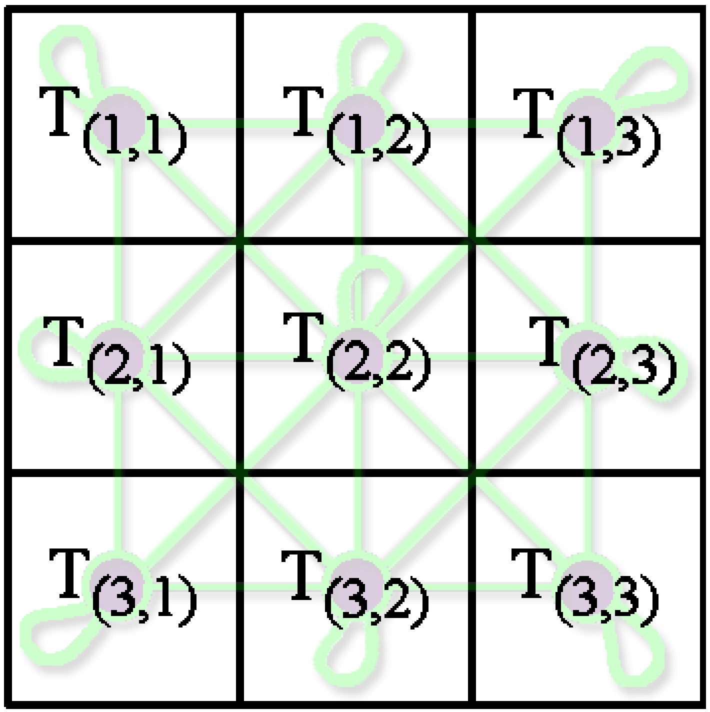
Which grid sizes are always solvable?

We can treat the puzzle grid as a graph





3×3 Adjacency Matrix…



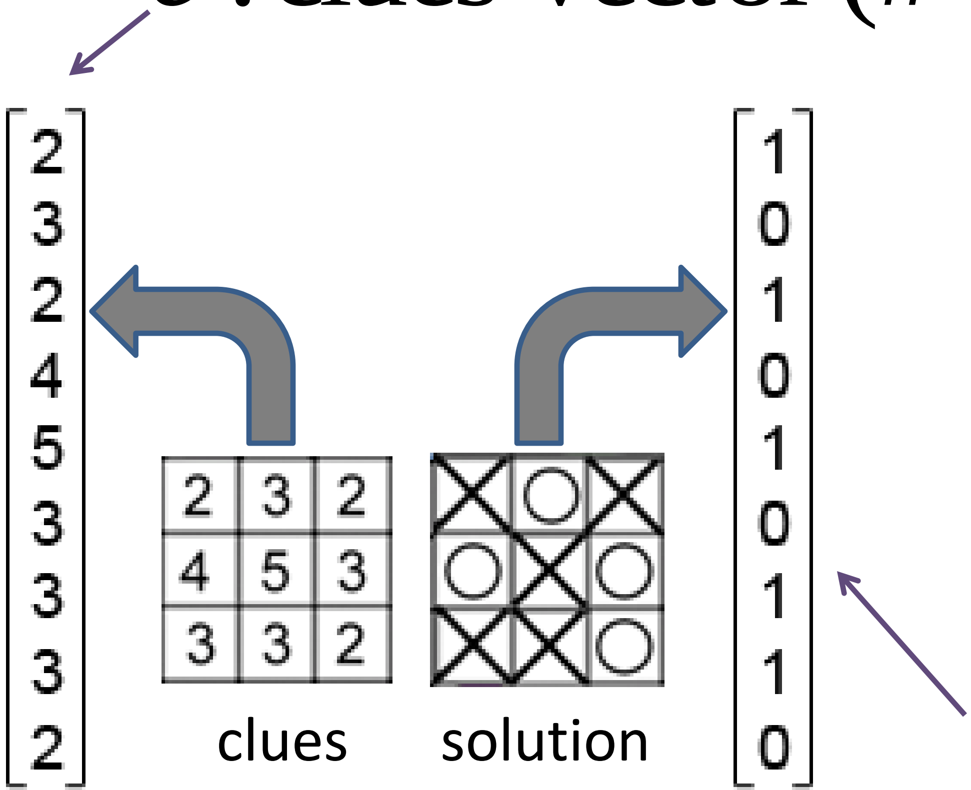
	$T_{(1,1)}$	$T_{(1,2)}$	$T_{(1,3)}$	$T_{(2,1)}$	$T_{(2,2)}$	$T_{(2,3)}$	$T_{(3,1)}$	$T_{(3,2)}$	$T_{(3,3)}$
$T_{(1,1)}$	1	1	0	1	1	0	0	0	0
$T_{(1,2)}$	1	1	1	1	1	1	0	0	0
$T_{(1,3)}$	0	1	1	0	1	1	0	0	0
$T_{(2,1)}$	1	1	0	1	1	0	1	1	0
$T_{(2,2)}$	1	1	1	1	1	1	1	1	1
$T_{(2,3)}$	0	1	1	0	1	1	0	1	1
$T_{(3,1)}$	0	0	0	1	1	0	1	1	0
$T_{(3,2)}$	0	0	0	1	1	1	1	1	1
$T_{(3,3)}$	0	0	0	0	1	1	0	1	1

T: locations in the grid

A: corresponding adjacency matrix

3×3 Solution and Clue Vectors…

$\vec{c}$: clues vector (#'s from 0 to 9)



$\vec{s}$: solution vector

(0's for O's and 1's for X's)

Solvability

creating puzzle…

$$A\vec{s} = \vec{c}$$

A: adjacency matrix

$\vec{s}$: solution vector

$\vec{c}$: clues vector

Solvability

creating puzzle…

$$A\vec{s} = \vec{c}$$

solving puzzle…

$$\vec{s} = A^{-1}\vec{c}$$

$A:$ adjacency matrix

$\vec{s}:$ solution vector

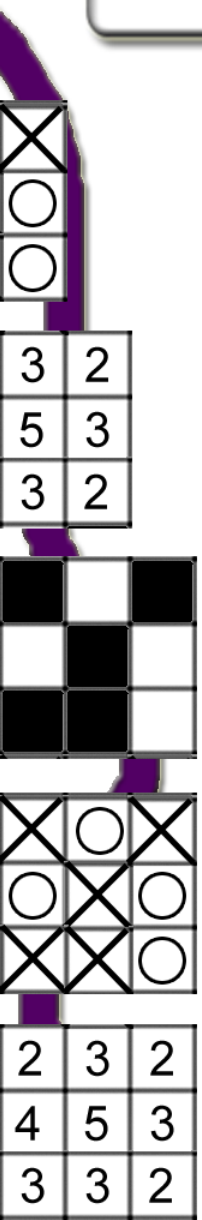
$\vec{c}:$ clues vector

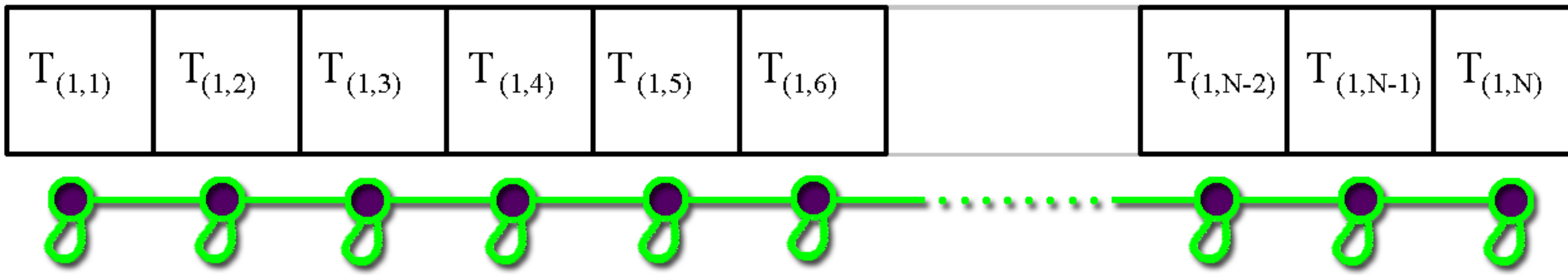
If the corresponding adjacency matrix, A, is invertible, then the puzzle is solvable!

When is the 1×N Puzzle Solvable?

- Our goal is to discover when an M×N matrix puzzle is solvable, but let’s solve a simpler problem first.

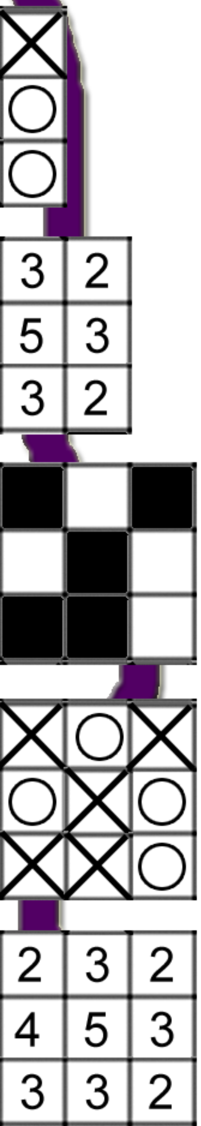
When is the 1×N Puzzle Solvable?





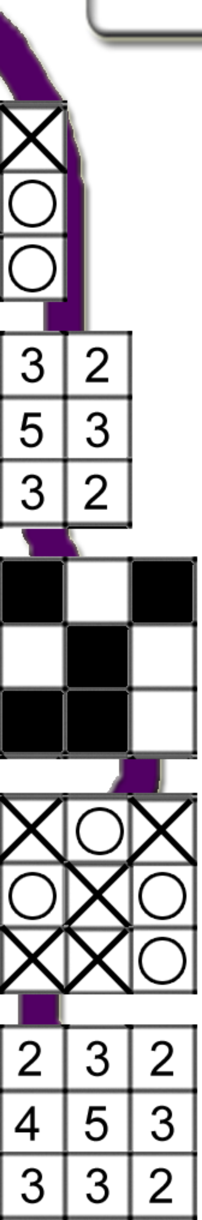
$$
\begin{array}{c|cccccccc}
 & T_{(1,1)} & T_{(1,2)} & T_{(1,3)} & T_{(1,4)} & \cdots & & & T_{(1,N)} \\
\hline
T_{(1,1)} & 1 & 1 & 0 & 0 & . & . & . & 0 \\
T_{(1,2)} & 1 & 1 & 1 & 0 & & & & . \\
T_{(1,3)} & 0 & 1 & 1 & 1 & & & & . \\
T_{(1,4)} & 0 & 0 & 1 & . & & & & . \\
\vdots & . & & & & . & & & . \\
 & . & & & & & 1 & 1 & 0 \\
 & . & & & & & 1 & 1 & 1 \\
T_{(1,N)} & 0 & . & . & . & . & 0 & 1 & 1
\end{array}
$$

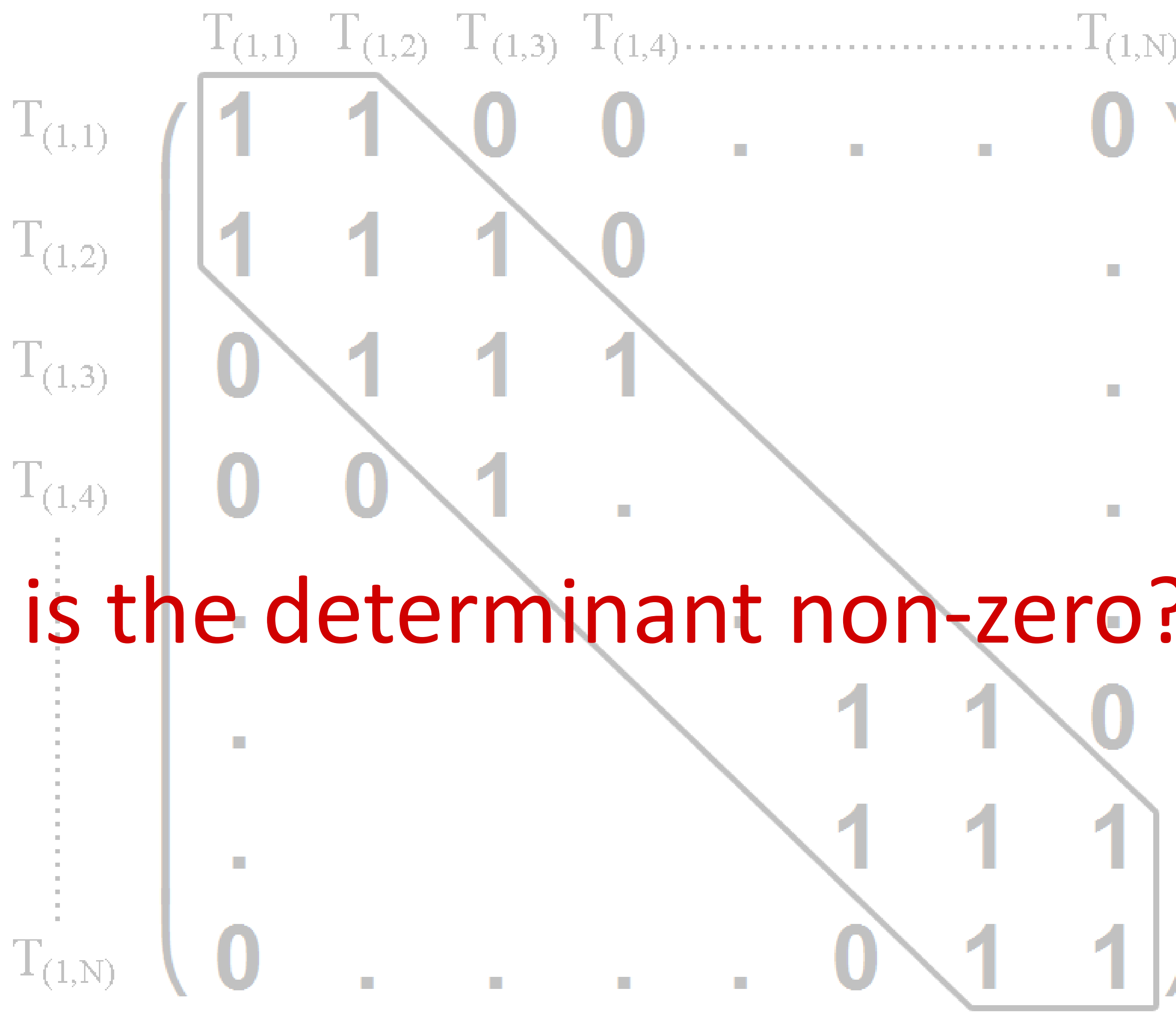
When is the 1×N Adjacency Matrix Invertible?



$$
\begin{array}{c|cccccccc}
 & T_{(1,1)} & T_{(1,2)} & T_{(1,3)} & T_{(1,4)} & \cdots & & & T_{(1,N)} \\
\hline
T_{(1,1)} & 1 & 1 & 0 & 0 & . & . & . & 0 \\
T_{(1,2)} & 1 & 1 & 1 & 0 & & & & . \\
T_{(1,3)} & 0 & 1 & 1 & 1 & & & & . \\
T_{(1,4)} & 0 & 0 & 1 & . & & & & . \\
\vdots & . & & & & . & & & . \\
 & . & & & & & 1 & 1 & 0 \\
 & . & & & & & 1 & 1 & 1 \\
T_{(1,N)} & 0 & . & . & . & . & 0 & 1 & 1
\end{array}
$$

When is the 1×N Adjacency Matrix Invertible?



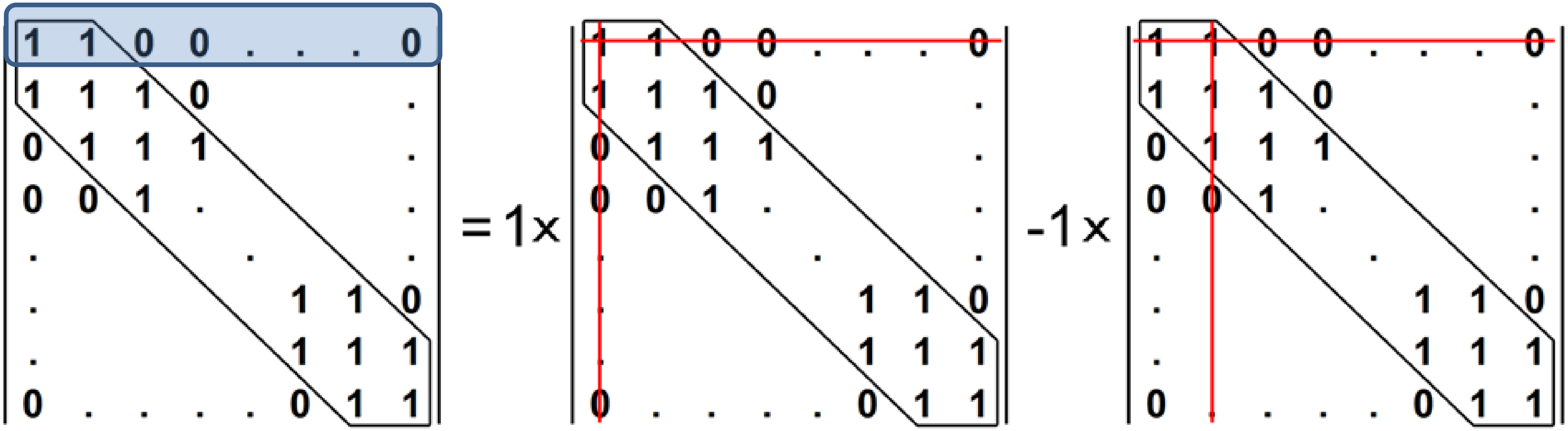


When is the determinant non-zero?

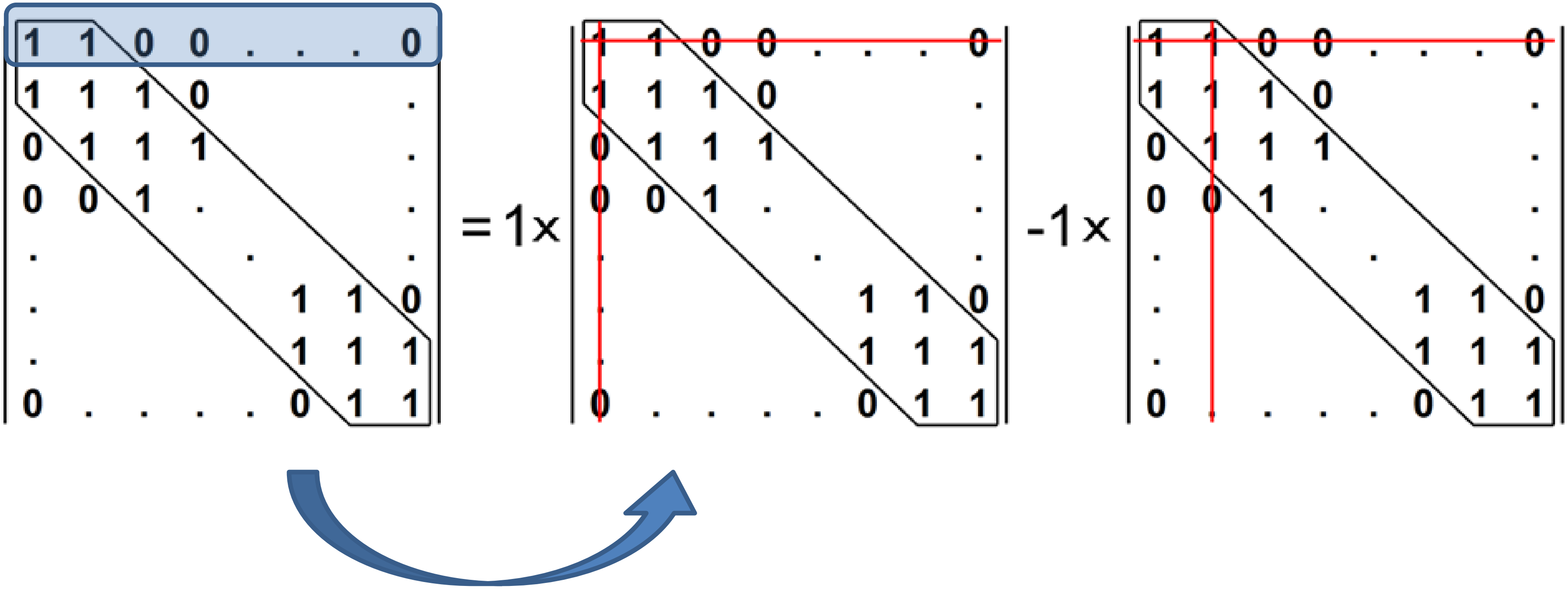
Determinant of a 1×k adjacency matrix..

$$\begin{vmatrix} 1 & 1 & 0 & 0 & . & . & . & 0 \\ 1 & 1 & 1 & 0 & & & & . \\ 0 & 1 & 1 & 1 & & & & . \\ 0 & 0 & 1 & . & & & & . \\ . & & & & . & & & . \\ . & & & & & 1 & 1 & 0 \\ . & & & & & 1 & 1 & 1 \\ 0 & . & . & . & . & 0 & 1 & 1 \end{vmatrix} =$$

Determinant of a 1×k adjacency matrix..

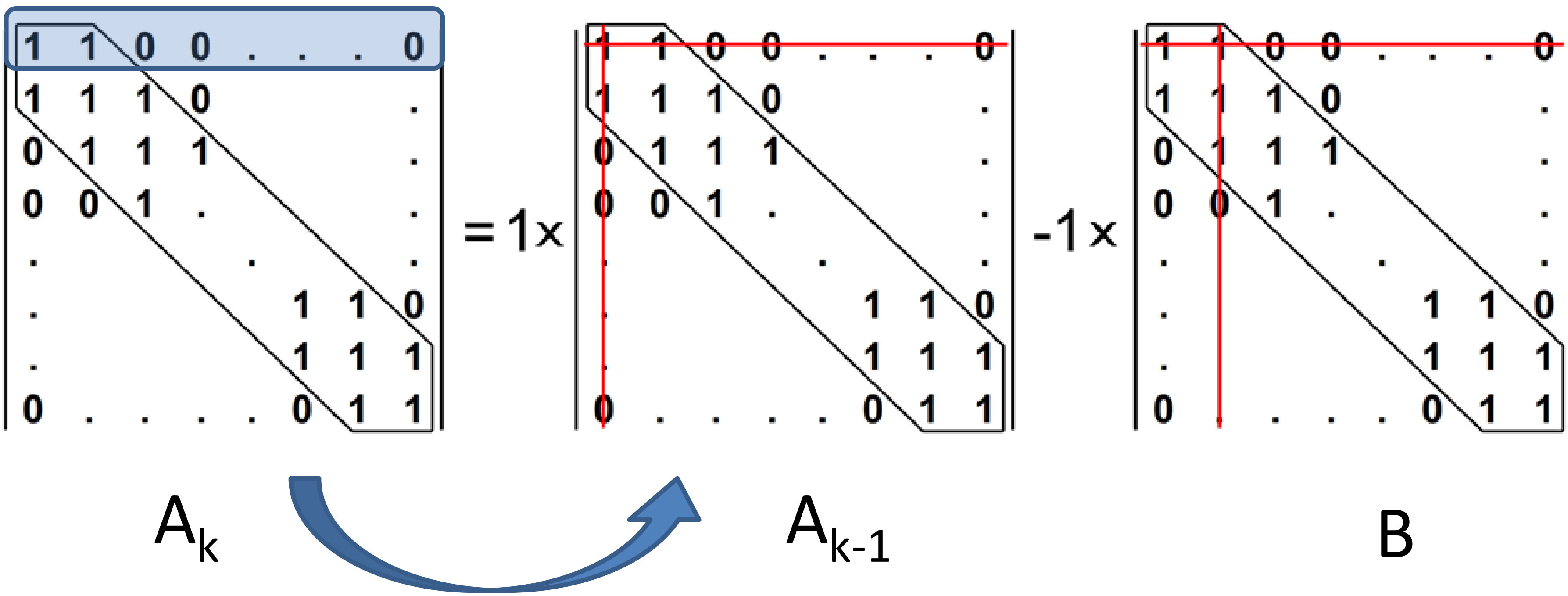


Determinant of a 1×k adjacency matrix..



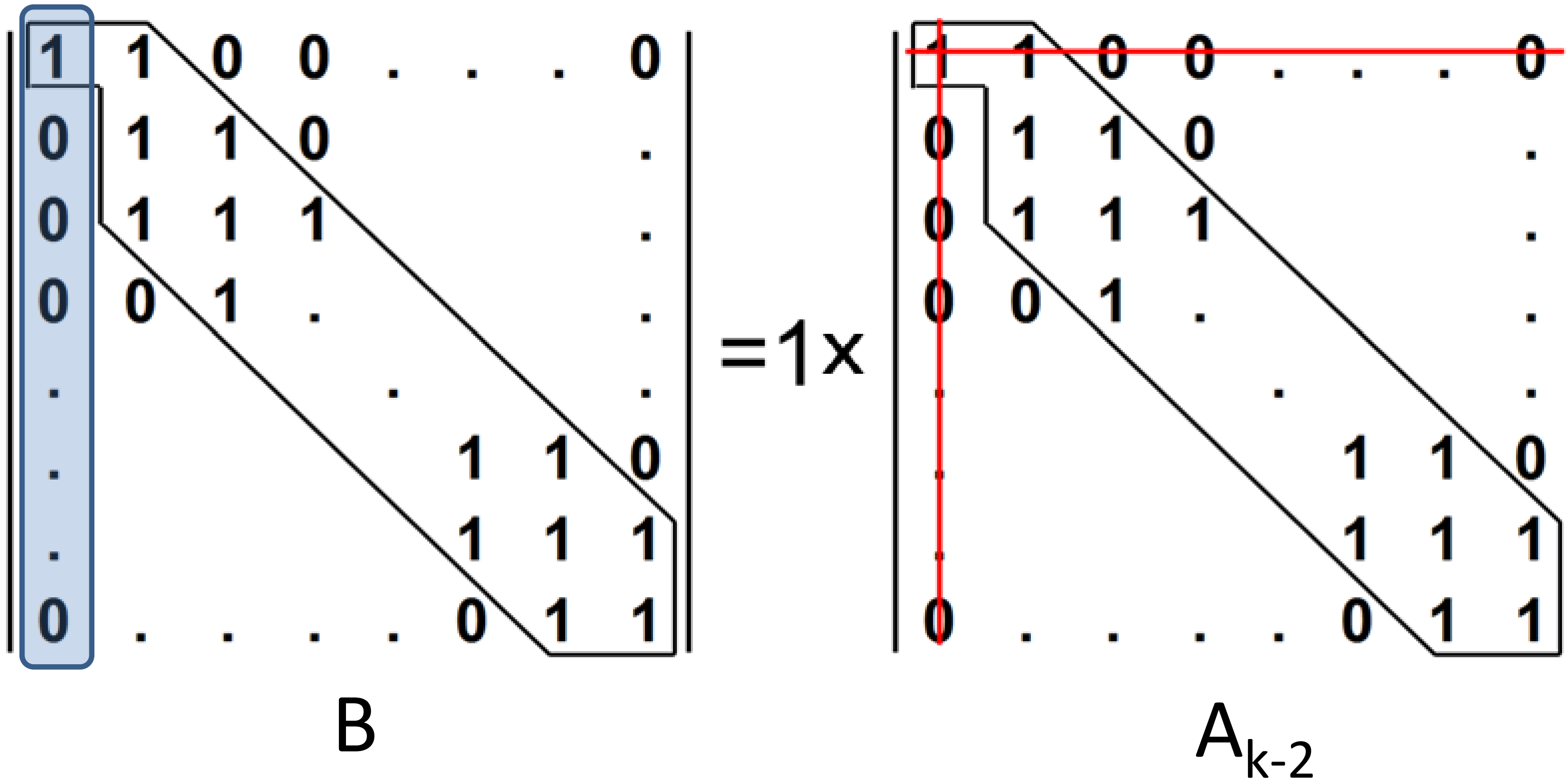
The same form of a matrix, but with a
different number of rows and columns

Determinant of a 1×k adjacency matrix..



The same form of a matrix, but with a different number of rows and columns

Expanding again…



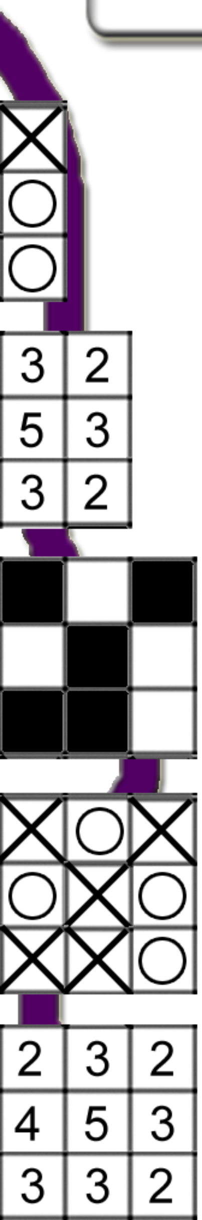
Expanding by the first column.

Determinant of a 1×k adjacency matrix..

$$\underbrace{\begin{vmatrix} 1 & 1 & 0 & 0 & . & . & . & 0 \\ 1 & 1 & 1 & 0 & & & & . \\ 0 & 1 & 1 & 1 & & & & . \\ 0 & 0 & 1 & . & & & & . \\ . & & & & . & & & . \\ . & & & & & 1 & 1 & 0 \\ . & & & & & 1 & 1 & 1 \\ 0 & . & . & . & . & 0 & 1 & 1 \end{vmatrix}}_{A_k} = 1\times \underbrace{\begin{vmatrix} 1 & 1 & 0 & 0 & . & . & . & 0 \\ 1 & 1 & 1 & 0 & & & & . \\ 0 & 1 & 1 & 1 & & & & . \\ 0 & 0 & 1 & . & & & & . \\ . & & & & . & & & . \\ . & & & & & 1 & 1 & 0 \\ . & & & & & 1 & 1 & 1 \\ 0 & . & . & . & . & 0 & 1 & 1 \end{vmatrix}}_{A_{k-1}} - 1\times \underbrace{\begin{vmatrix} 1 & 1 & 0 & 0 & . & . & . & 0 \\ 1 & 1 & 1 & 0 & & & & . \\ 0 & 1 & 1 & 1 & & & & . \\ 0 & 0 & 1 & . & & & & . \\ . & & & & . & & & . \\ . & & & & & 1 & 1 & 0 \\ . & & & & & 1 & 1 & 1 \\ 0 & . & . & . & . & 0 & 1 & 1 \end{vmatrix}}_{A_{k-2}}$$

$$\det(A_k)=\det(A_{k-1})-\det(A_{k-2})$$

A little algebra…



$$\det(A_k)=\det(A_{k-1})-\det(A_{k-2})$$

A little algebra…

$$\det(A_k)=\det(A_{k-1})-\det(A_{k-2})$$

$$\det(A_{k-1})=\det(A_{k-2})-\det(A_{k-3})$$

A little algebra…

$$\det(A_k)=\det(A_{k-1})-\det(A_{k-2})$$

$$\det(A_{k-1})=\det(A_{k-2})-\det(A_{k-3})$$

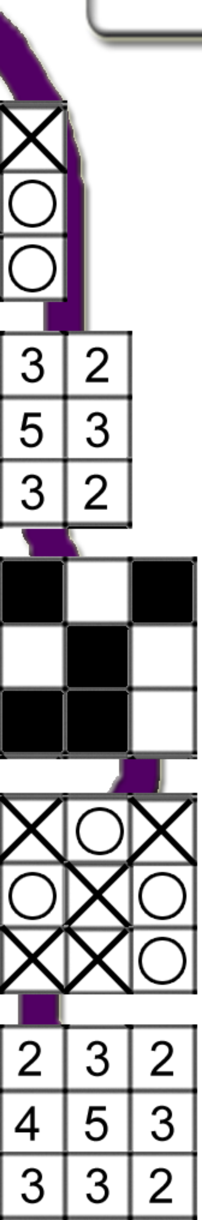
$$\det(A_k)=\det(A_{k-2})-\det(A_{k-3})-\det(A_{k-2})$$

A little algebra…

$$\det(A_k)=\det(A_{k-1})-\det(A_{k-2})$$

$$\det(A_{k-1})=\det(A_{k-2})-\det(A_{k-3})$$

$$\det(A_k)=\cancel{\det(A_{k-2})}-\det(A_{k-3})-\cancel{\det(A_{k-2})}$$



A little algebra…

$$\det(A_k)=\det(A_{k-1})-\det(A_{k-2})$$

$$\det(A_{k-1})=\det(A_{k-2})-\det(A_{k-3})$$

$$\boxed{\det(A_k)=-\det(A_{k-3})}$$

Proof by Strong Induction…

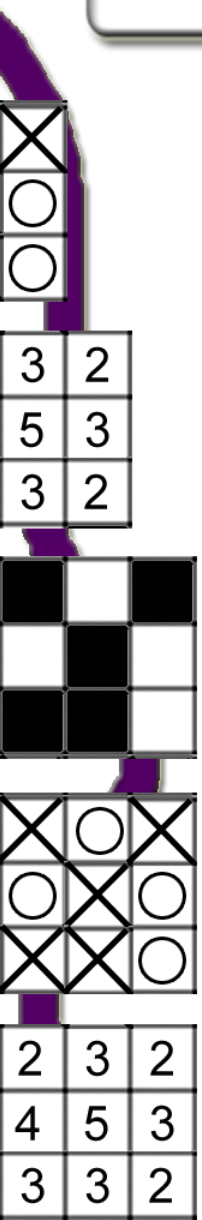
$$\det(A_k) = \begin{cases} 1\times(-1)^{k+1} & \text{if } k \equiv 1 \pmod{3} \\ 0 & \text{if } k \equiv 2 \pmod{3} \\ 1\times(-1)^{k} & \text{if } k \equiv 0 \pmod{3} \end{cases}$$

Base cases:	Given:
$\det(A_1)=1$	$\det(A_k)=-\det(A_{k-3})$
$\det(A_2)=0$	
$\det(A_3)=-1$	

Lemma

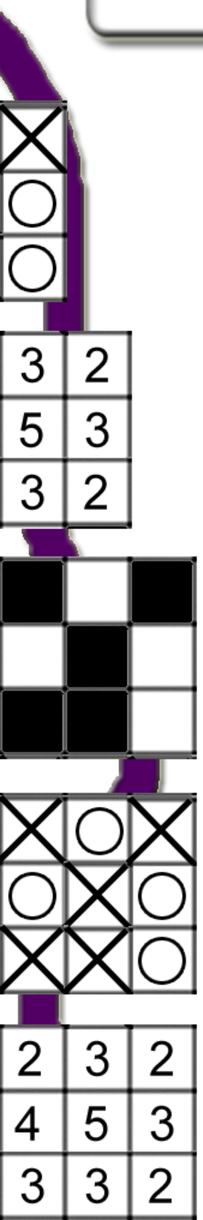
1×N puzzles are solvable iff $N \neq 2 \pmod{3}$

When is M×N Solvable?



Now that we've solved the 1×N case, let's solve the more general M×N case!

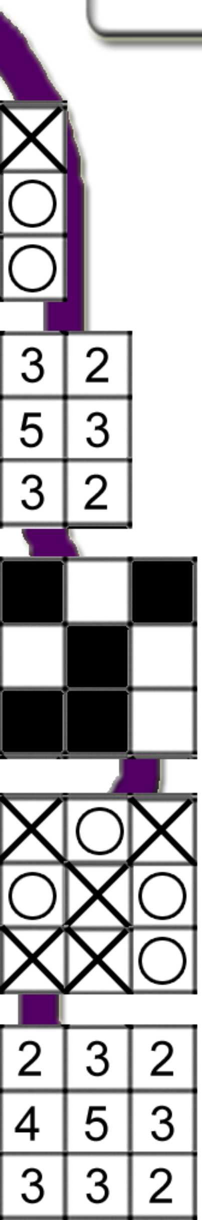
Is the 3×3 Solvable?



Yes, row reducing this matrix yields the identity

	$T_{(1,1)}$	$T_{(1,2)}$	$T_{(1,3)}$	$T_{(2,1)}$	$T_{(2,2)}$	$T_{(2,3)}$	$T_{(3,1)}$	$T_{(3,2)}$	$T_{(3,3)}$
$T_{(1,1)}$	1	1	0	1	1	0	0	0	0
$T_{(1,2)}$	1	1	1	1	1	1	0	0	0
$T_{(1,3)}$	0	1	1	0	1	1	0	0	0
$T_{(2,1)}$	1	1	0	1	1	0	1	1	0
$T_{(2,2)}$	1	1	1	1	1	1	1	1	1
$T_{(2,3)}$	0	1	1	0	1	1	0	1	1
$T_{(3,1)}$	0	0	0	1	1	0	1	1	0
$T_{(3,2)}$	0	0	0	1	1	1	1	1	1
$T_{(3,3)}$	0	0	0	0	1	1	0	1	1

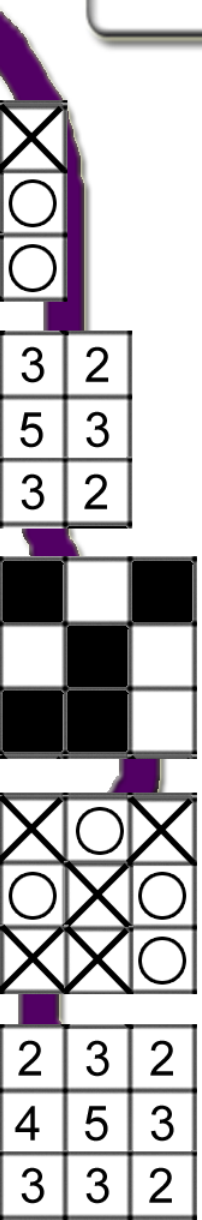
Looking for patterns…



The same 3×3 block is repeated

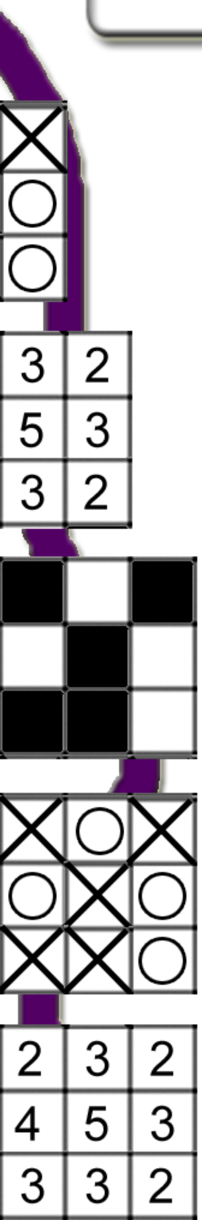
	$T_{(1,1)}$	$T_{(1,2)}$	$T_{(1,3)}$	$T_{(2,1)}$	$T_{(2,2)}$	$T_{(2,3)}$	$T_{(3,1)}$	$T_{(3,2)}$	$T_{(3,3)}$
$T_{(1,1)}$	1	1	0	1	1	0	0	0	0
$T_{(1,2)}$	1	1	1	1	1	1	0	0	0
$T_{(1,3)}$	0	1	1	0	1	1	0	0	0
$T_{(2,1)}$	1	1	0	1	1	0	1	1	0
$T_{(2,2)}$	1	1	1	1	1	1	1	1	1
$T_{(2,3)}$	0	1	1	0	1	1	0	1	1
$T_{(3,1)}$	0	0	0	1	1	0	1	1	0
$T_{(3,2)}$	0	0	0	1	1	1	1	1	1
$T_{(3,3)}$	0	0	0	0	1	1	0	1	1

The blocks are in fact the 1×3 A matrix



	$T_{(1,1)}$	$T_{(1,2)}$	$T_{(1,3)}$	$T_{(2,1)}$	$T_{(2,2)}$	$T_{(2,3)}$	$T_{(3,1)}$	$T_{(3,2)}$	$T_{(3,3)}$
$T_{(1,1)}$	1	1	0	1	1	0	0	0	0
$T_{(1,2)}$	1	1x3	1	1	1x3	1	0	0's	0
$T_{(1,3)}$	0	1	1	0	1	1	0	0	0
$T_{(2,1)}$	1	1	0	1	1	0	1	1	0
$T_{(2,2)}$	1	1x3	1	1	1x3	1	1	1x3	1
$T_{(2,3)}$	0	1	1	0	1	1	0	1	1
$T_{(3,1)}$	0	0	0	1	1	0	1	1	0
$T_{(3,2)}$	0	0's	0	1	1x3	1	1	1x3	1
$T_{(3,3)}$	0	0	0	0	1	1	0	1	1

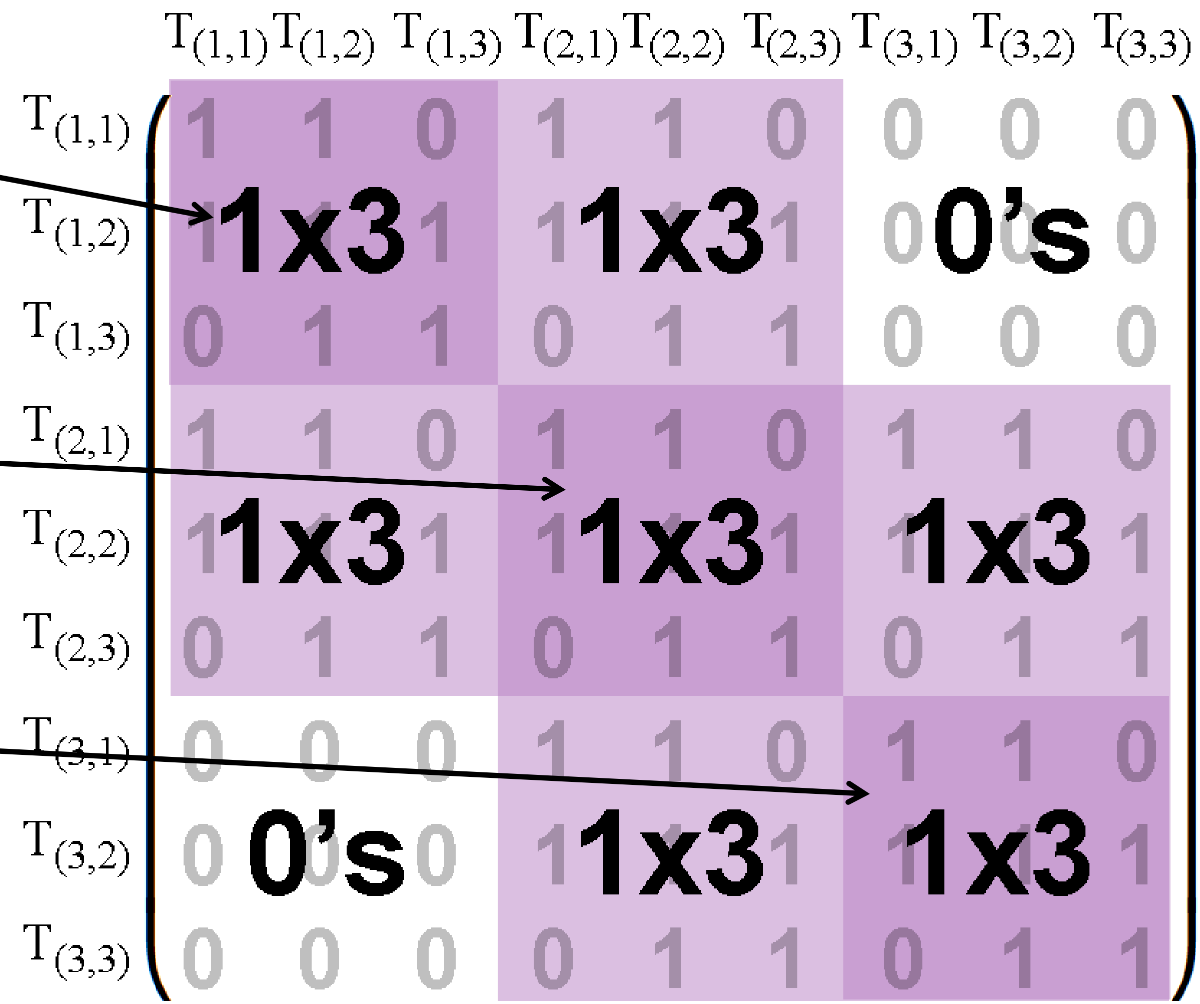
The blocks are in fact the 1×3 A matrix



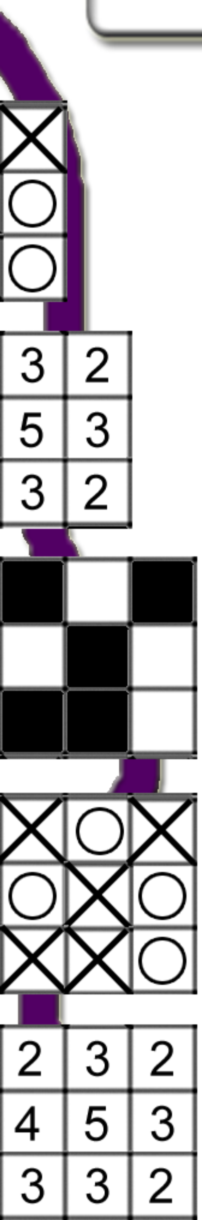
Upper row and how cells relate to each other

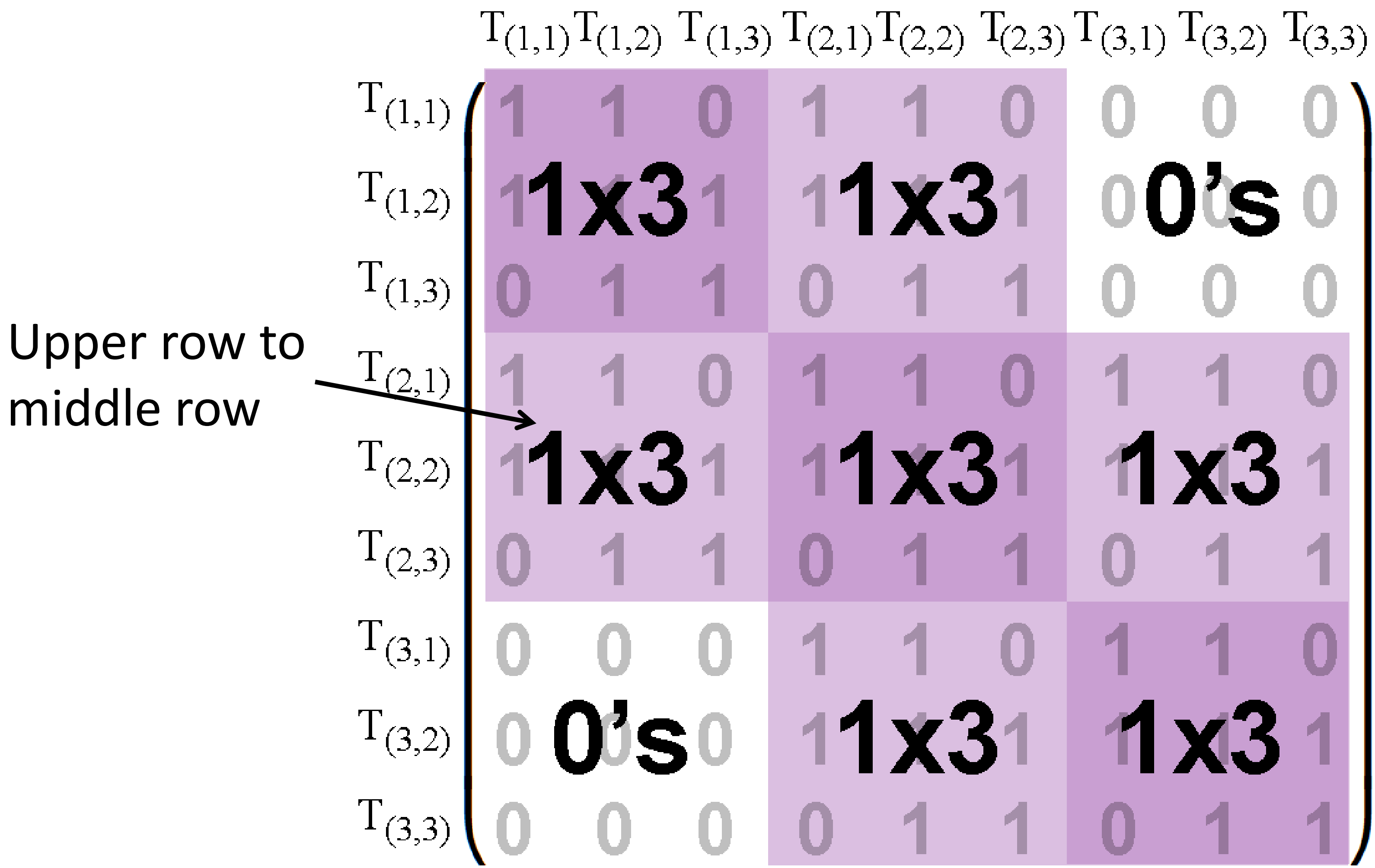
Middle row

Bottom row

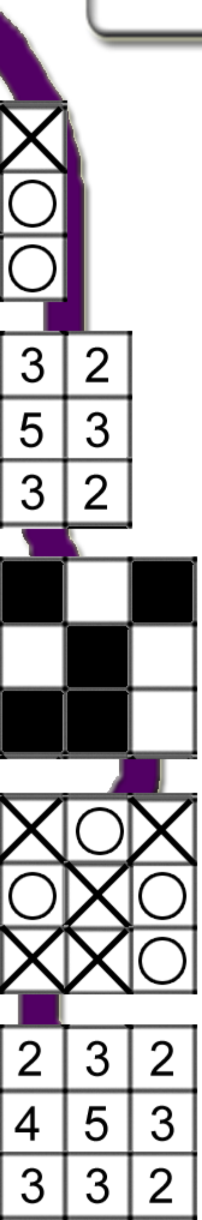


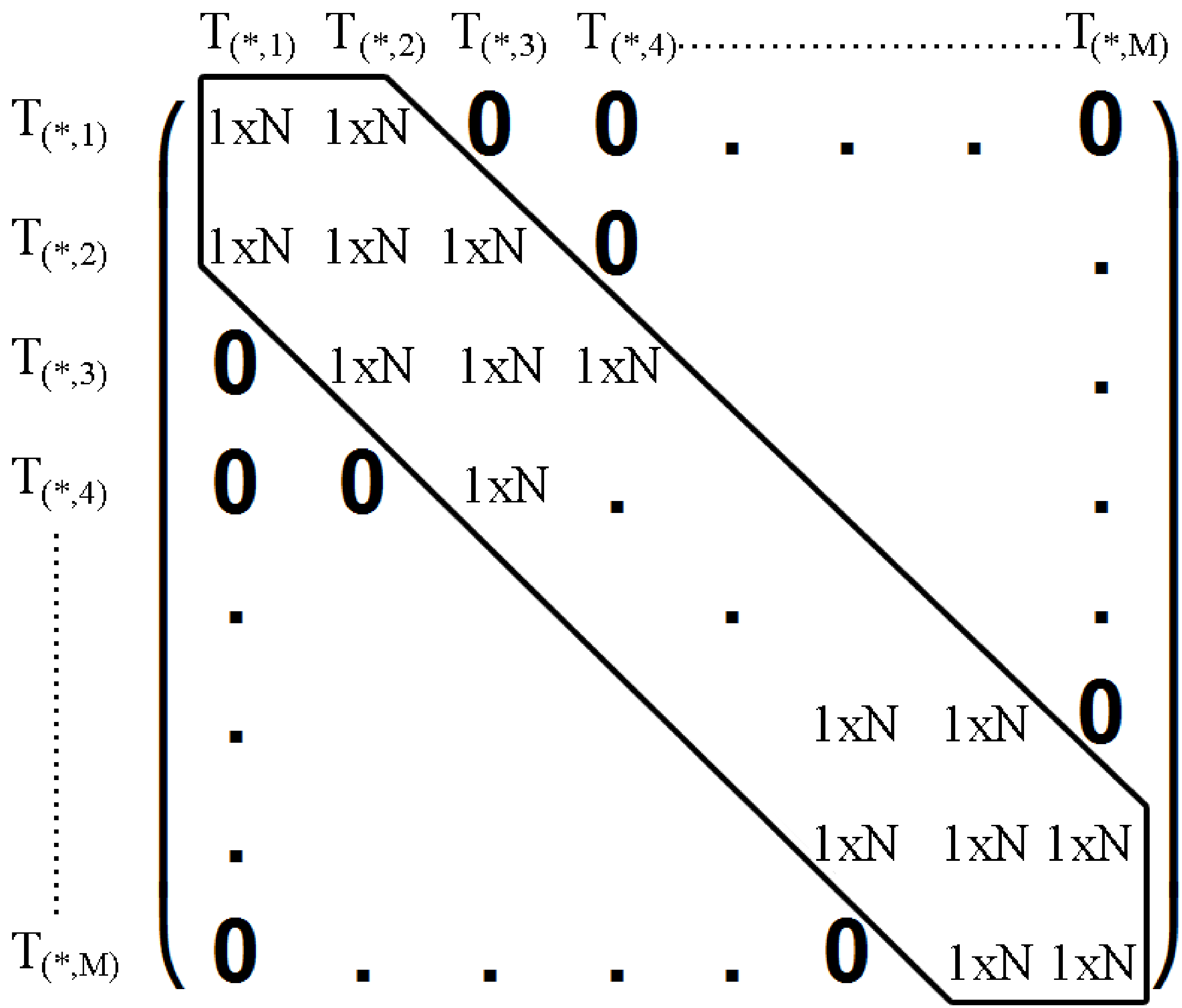
The blocks are in fact the 1×3 A matrix



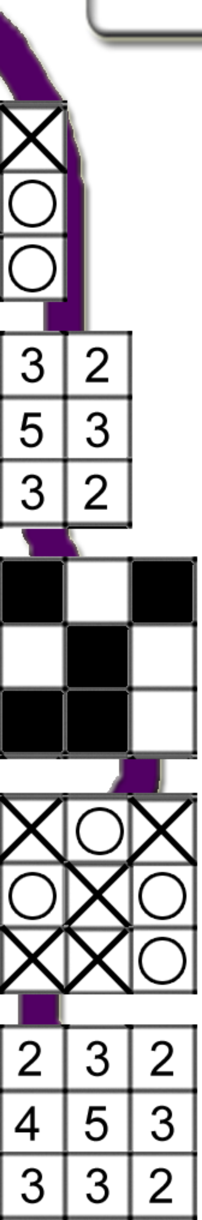


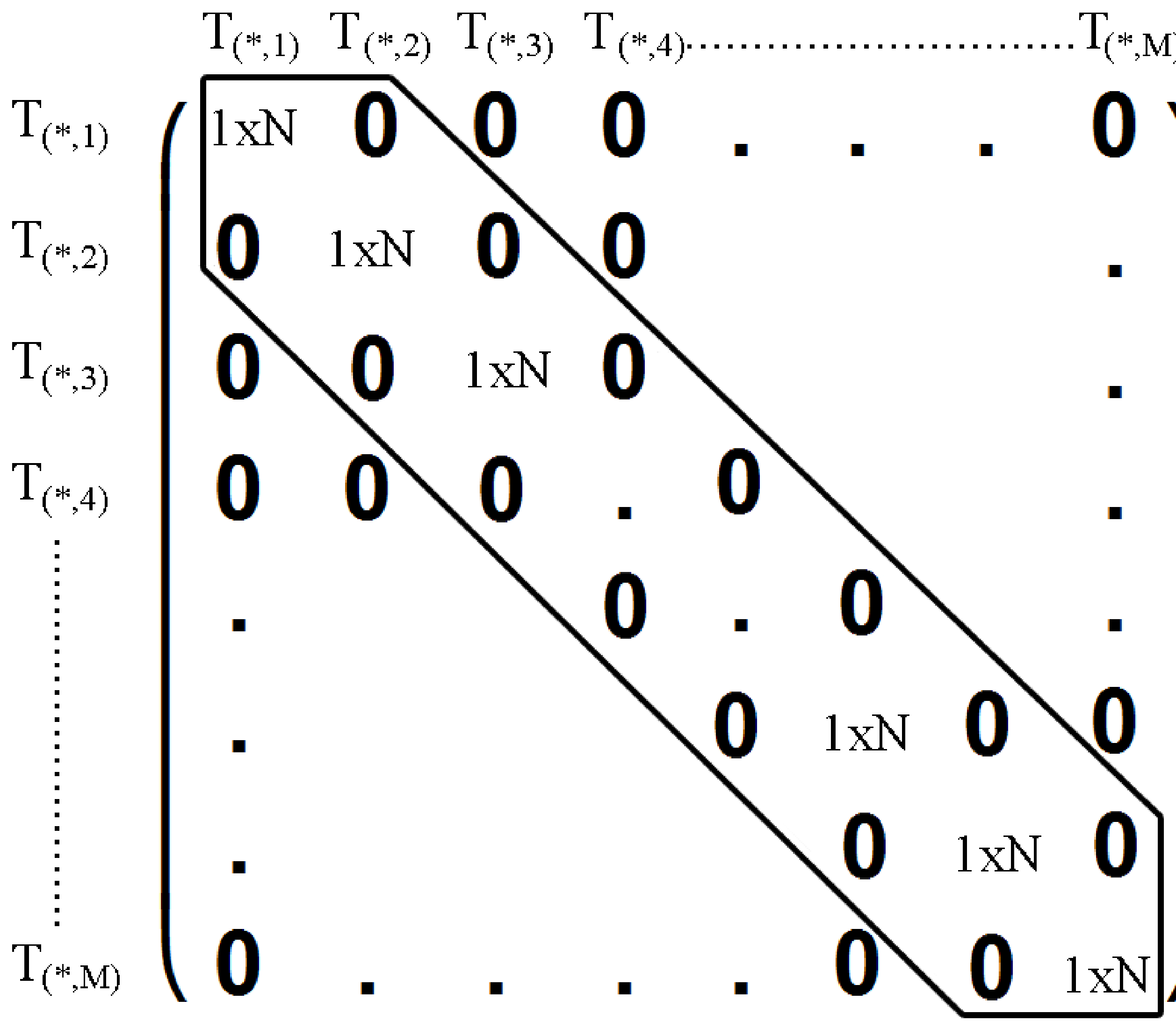
And the 1×N blocks for an M×N…



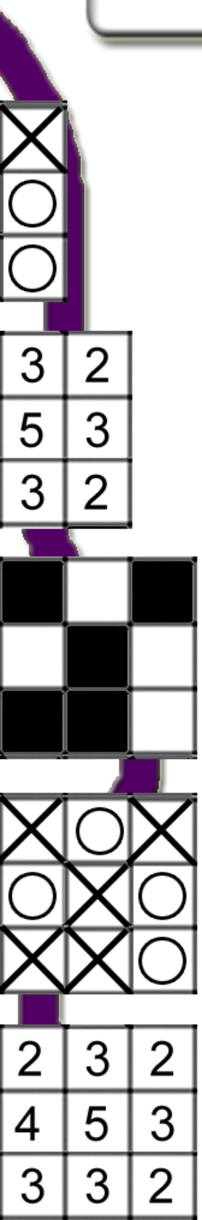


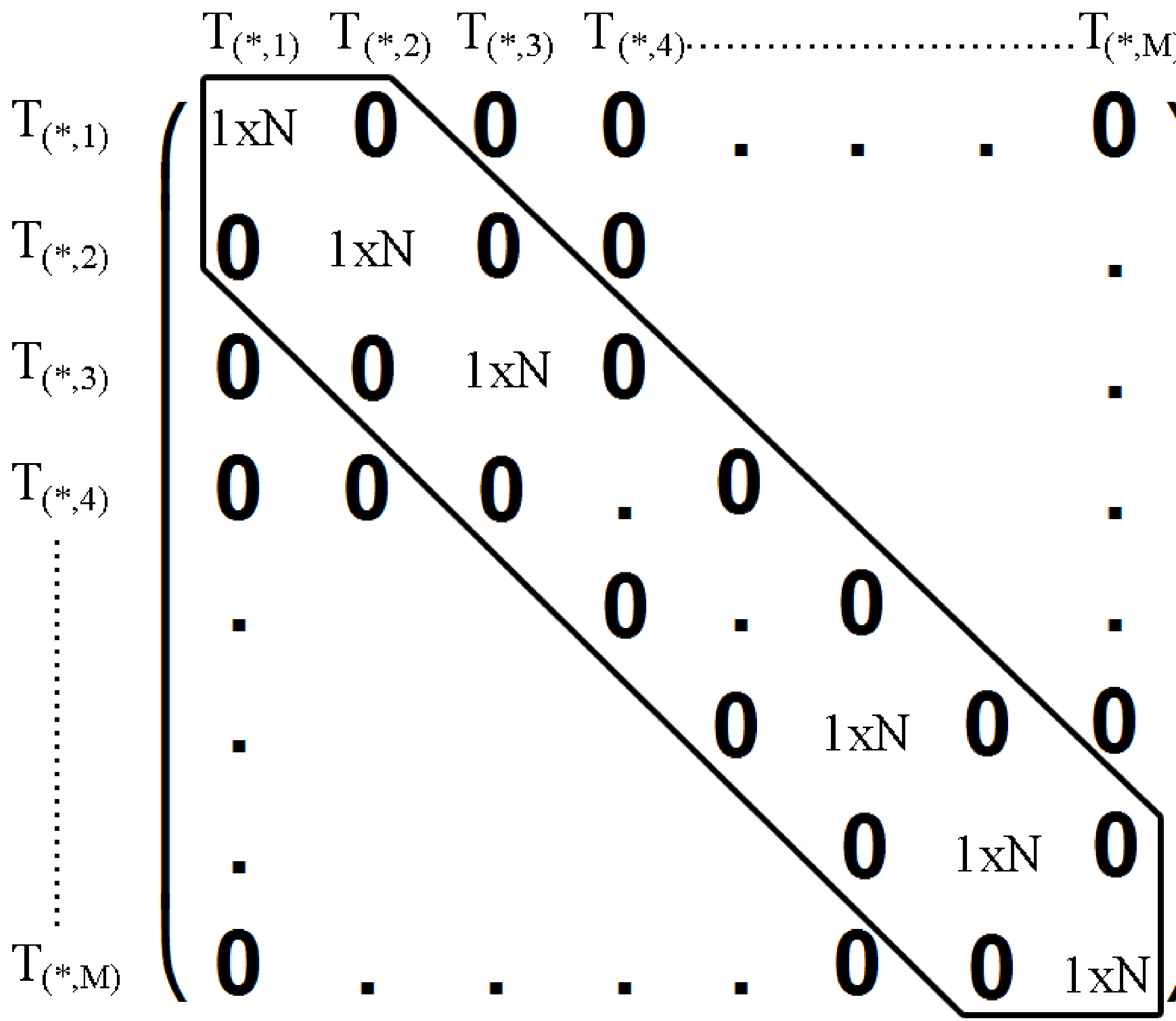
The M×N can be row reduced like this if the 1×M can be row reduced… [M ≠2 (mod 3)]



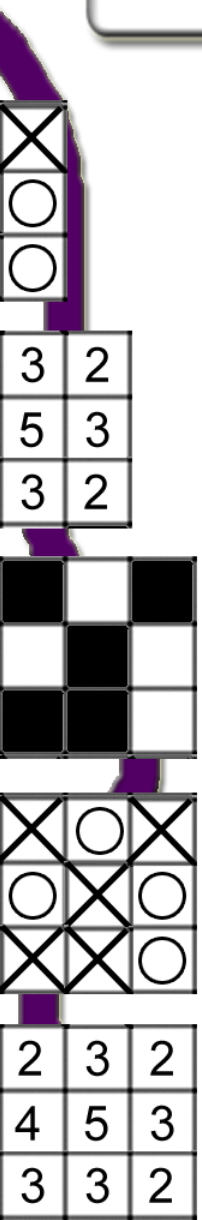


Each 1×N can be row reduced if N ≠2 (mod 3)





End of the Proof

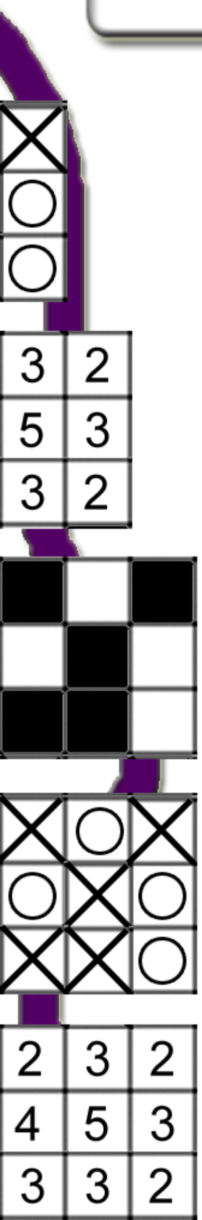


So any M×N puzzle is solvable if the 1×N and 1×M versions are solvable.

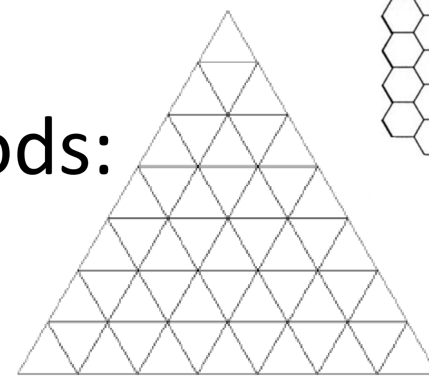
By our lemma, this is true whenever $N \neq 2 \pmod 3$ and $M \neq 2 \pmod 3$

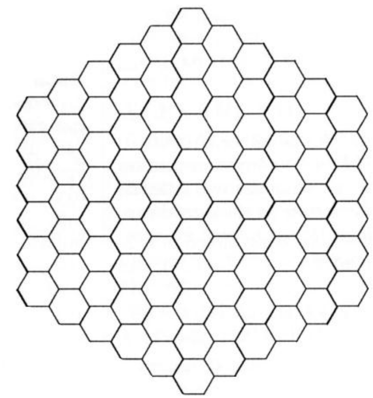
QED

Open Problems / Future Work

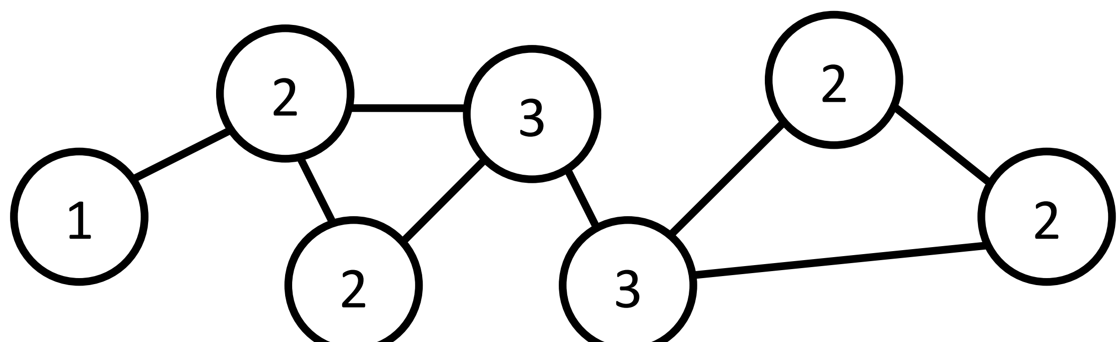


- Alternative lattice structures?
- Alternative neighborhoods:
  - e.g. don't count yourself

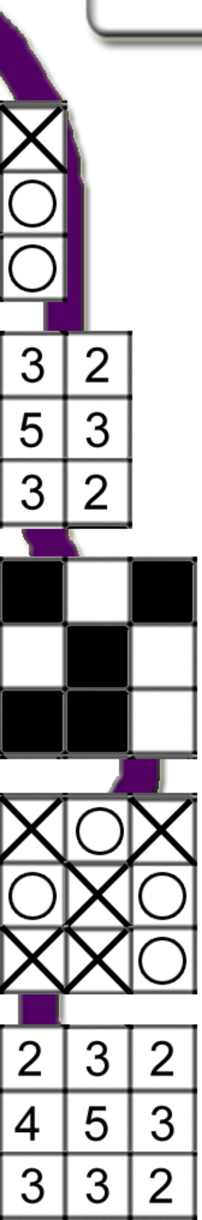




- Can we characterize when a puzzle is uniquely solvable when played on an arbitrary graph?



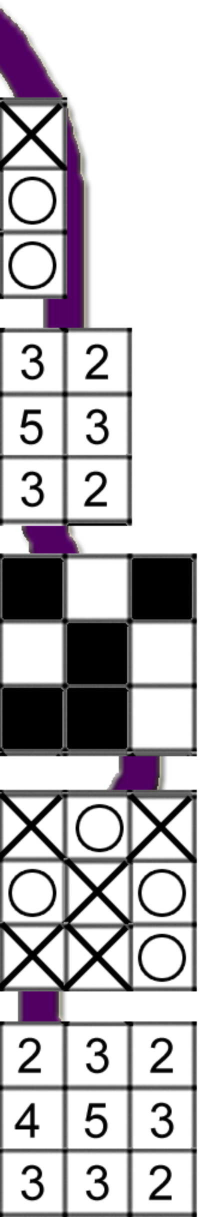
Take Think-Tac-Toe with you!



- Give the puzzles a try – they're fun!
- Give them to students
- Solve an open question from the previous slide
- Develop a new variation
- Something completely different

- And tell us about it – we'd love to hear from you!

http://www.stonedahl.com/thinktactoe/

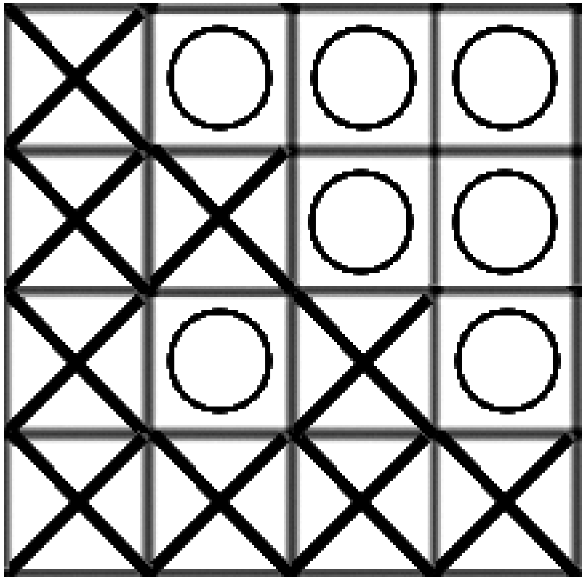


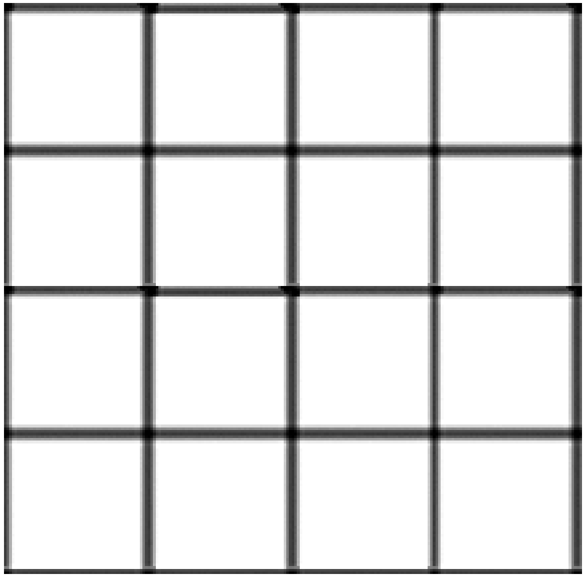
Questions?

Extra slides follow

Creating Puzzle…

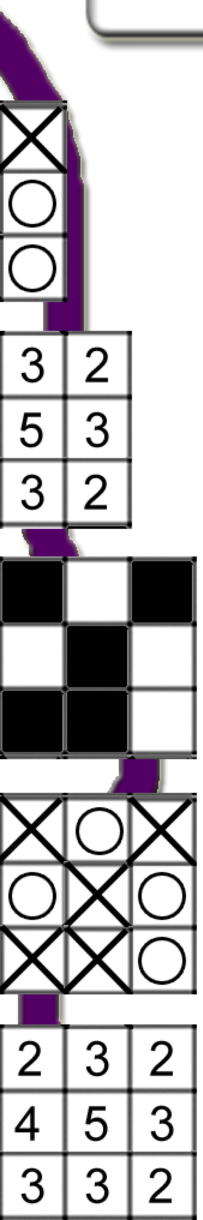
- Starting with the grid of X's and O's

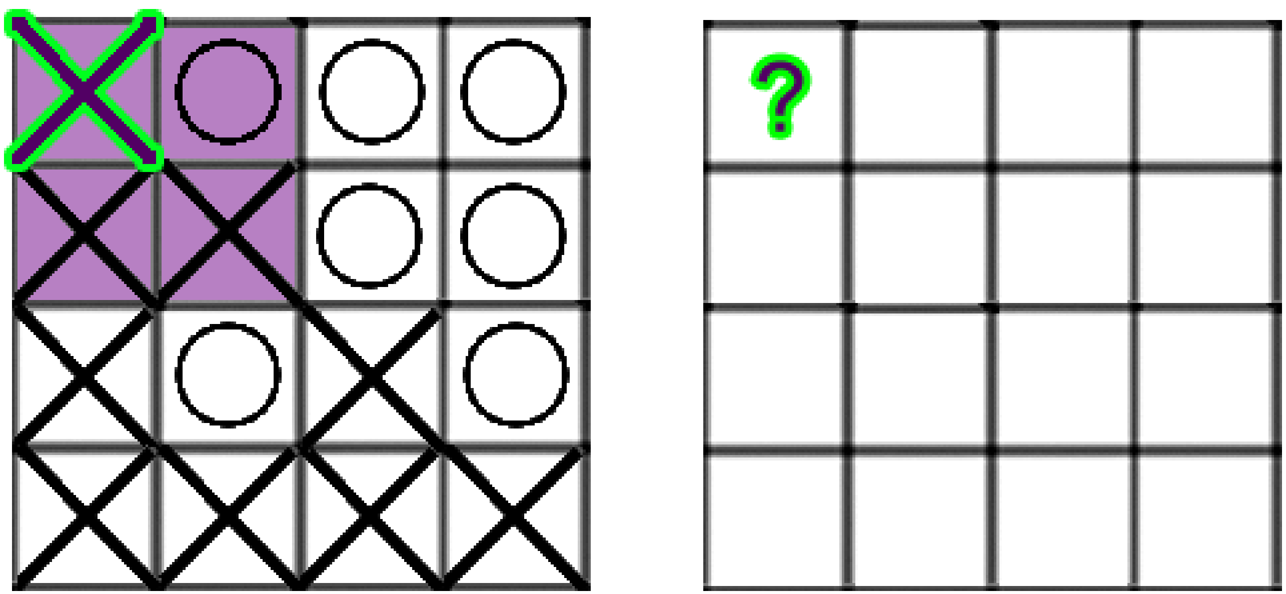




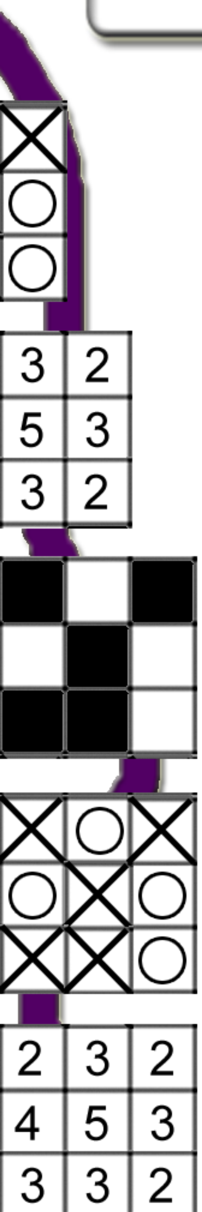
Creating Puzzle…

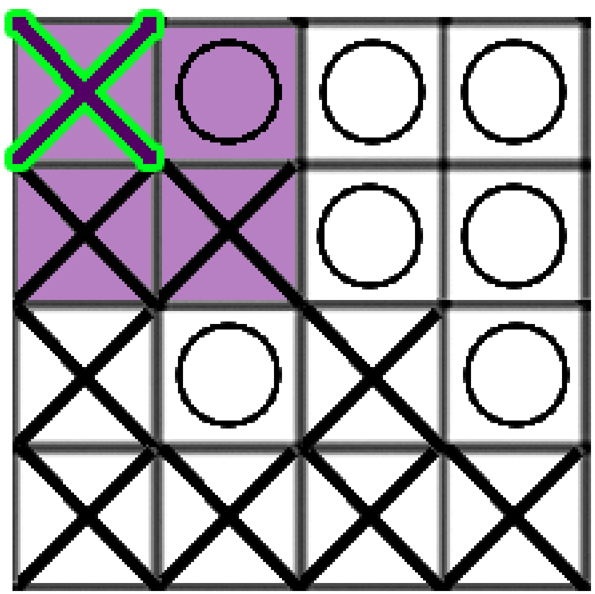
- How many X's are in the neighborhood?

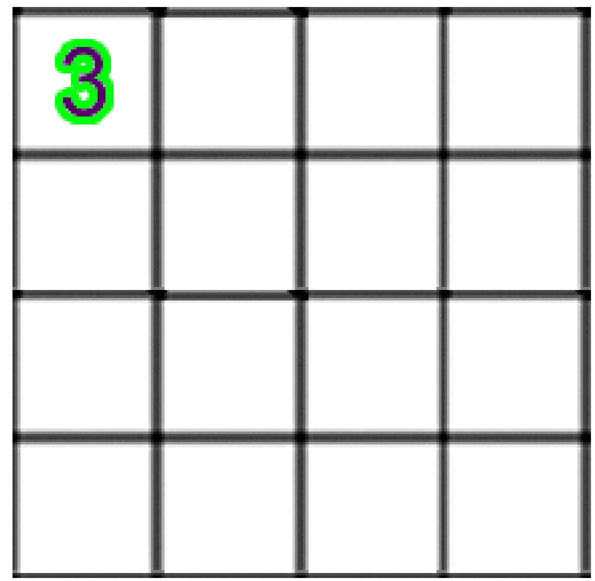




Creating Puzzle...

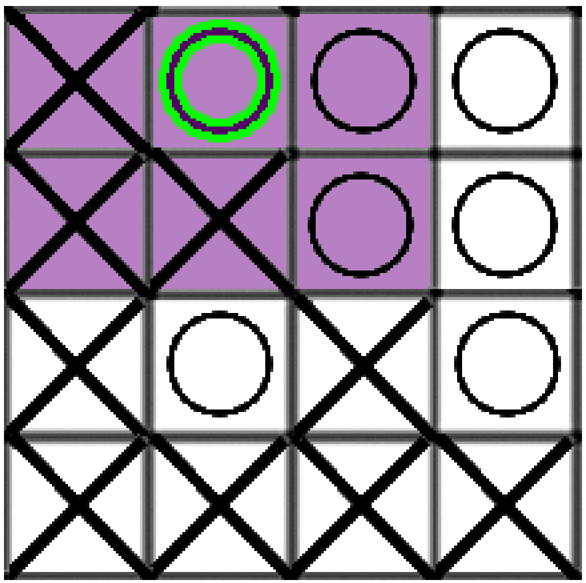


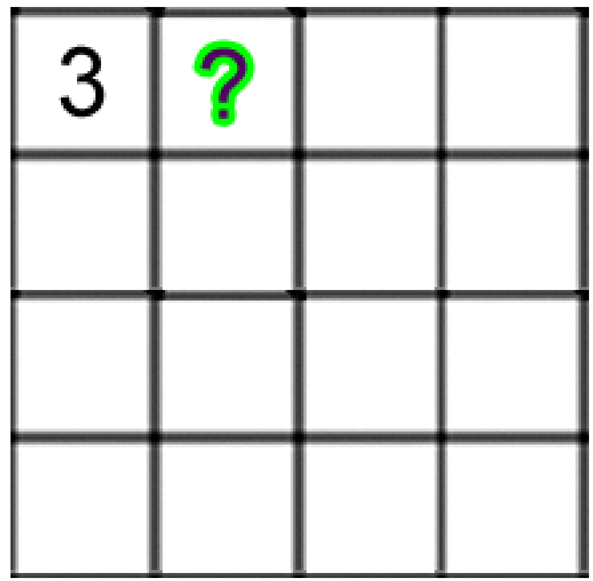


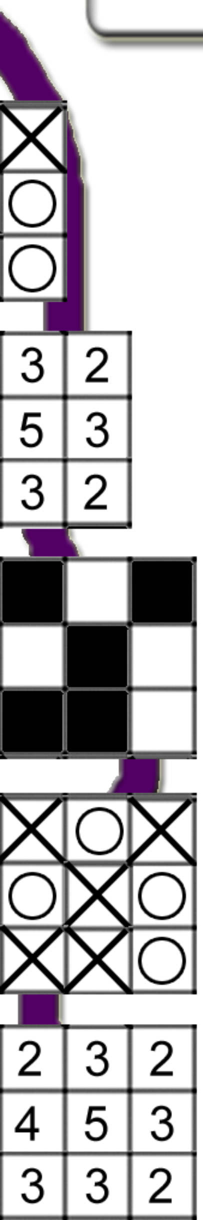


Creating Puzzle…

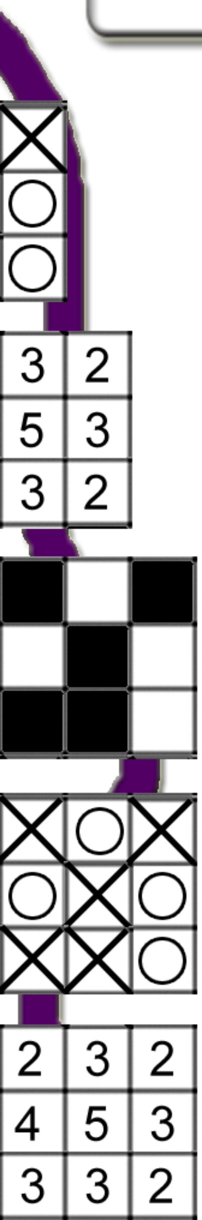
- How many X's are in this neighborhood?

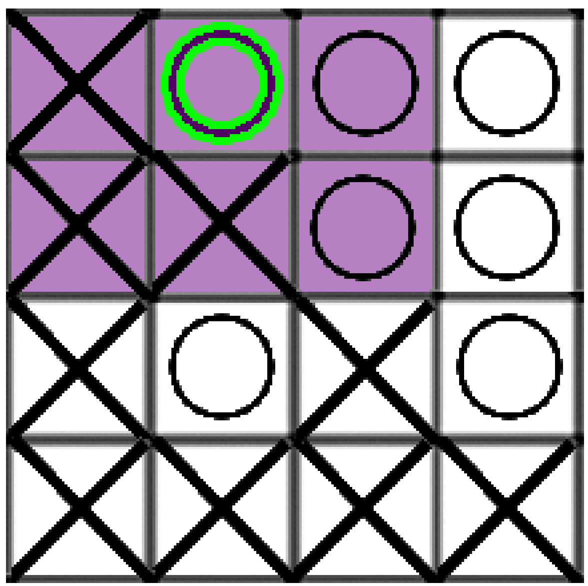


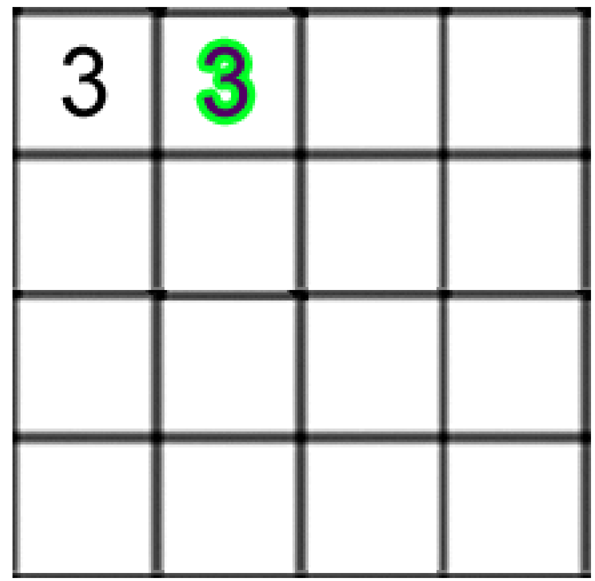


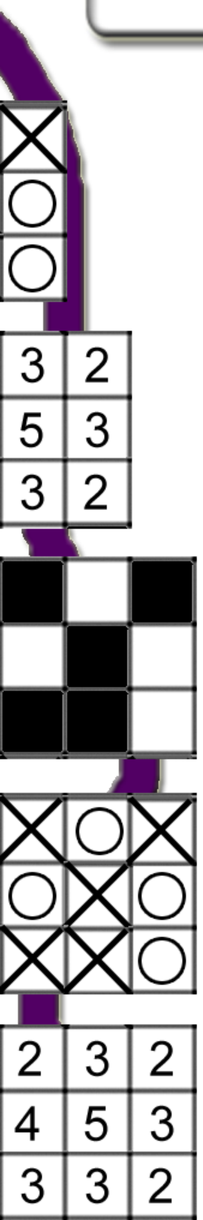


Creating Puzzle…



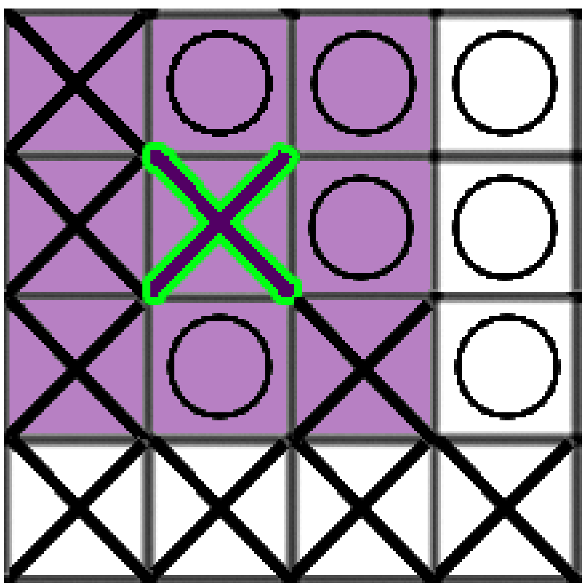


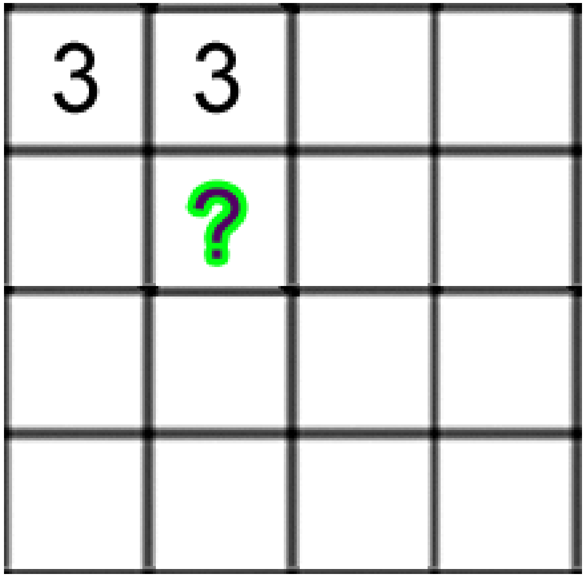




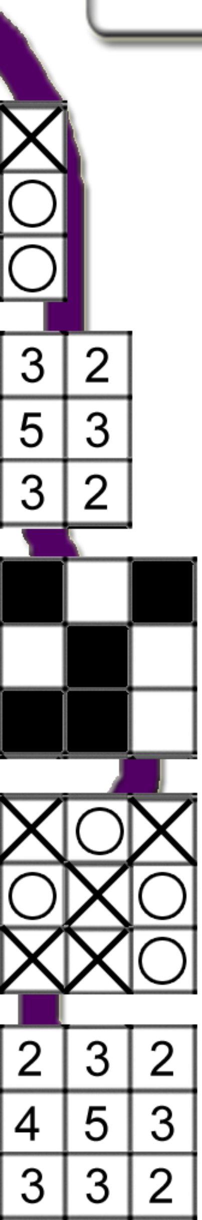
Creating Puzzle…

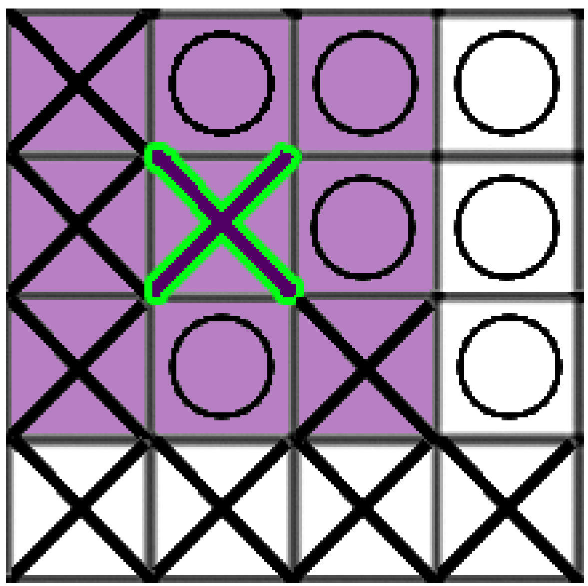
- And this neighborhood?

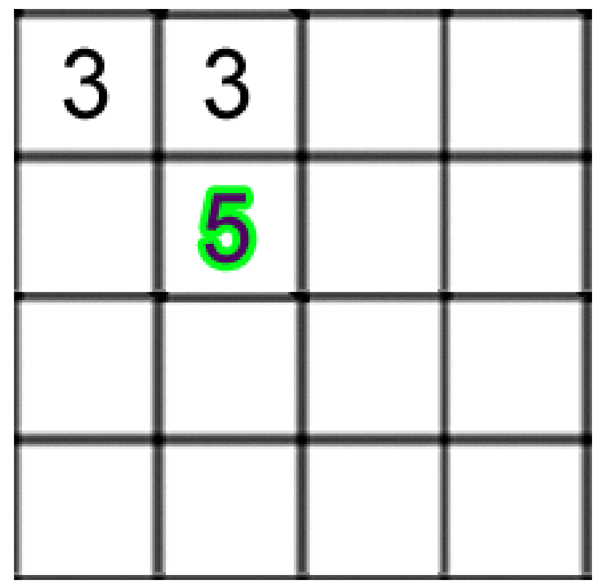




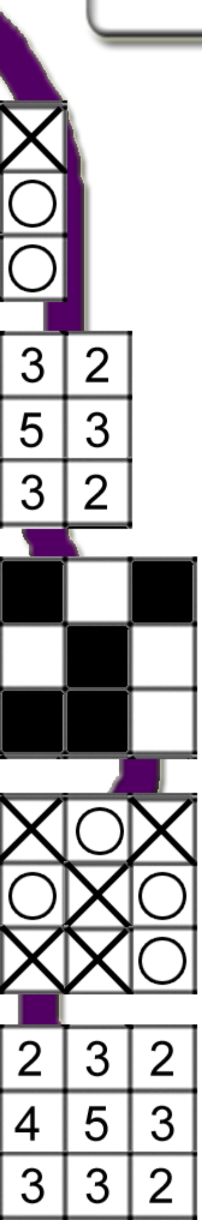
Creating Puzzle…



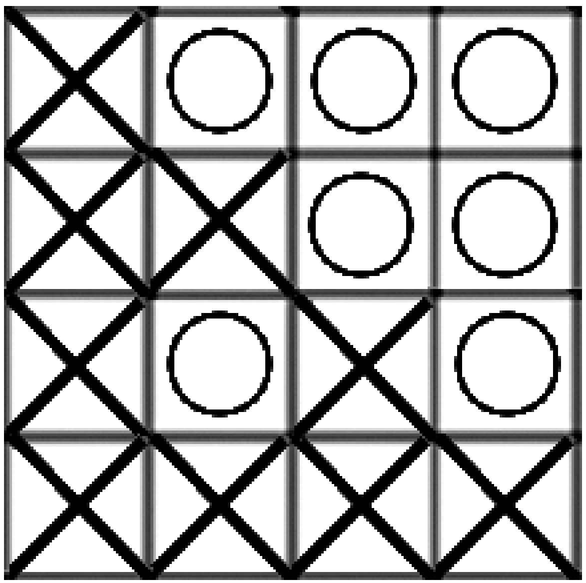




Creating Puzzle…



- So it's easy to create the puzzle,



3	3	1	0
4	5	2	1
5	7	5	3
3	5	4	3

but the fun part is solving it…

Counter example for 2 mod(3)

Given any NxM grid where N=2 mod(3), it is possible to fill the squares with x's and o's such that the clues are all 1's in at least two different ways as follows.

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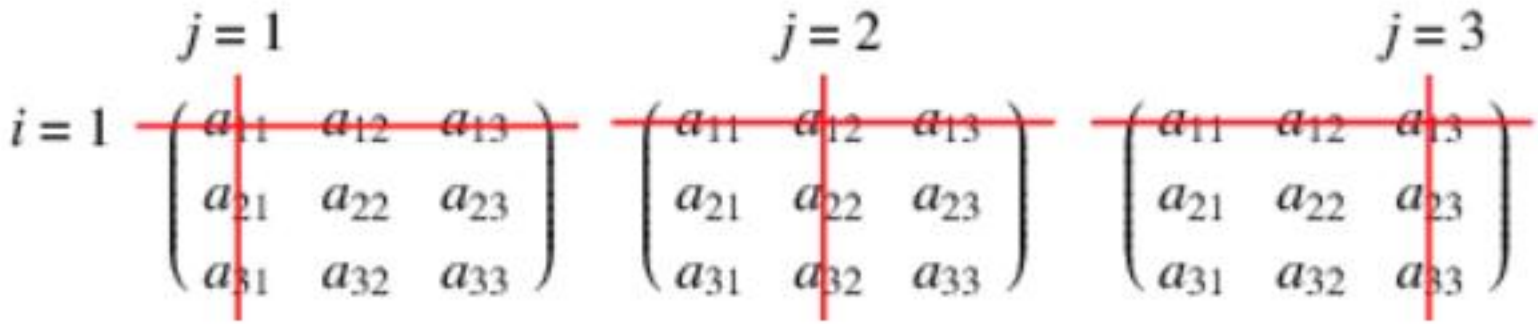
Using Expansion by Minors



For example, for a 3×3 matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(from Wolfram MathWorld)